Resummation of Soft Singularities and Improved Treatment of Heavy Quarks in Perturbative QCD

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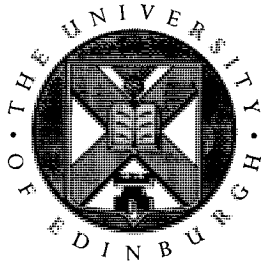

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Abstract

This thesis describes theoretical and numerical investigations into the resummation of soft singularities in some perturbative QCD cross sections. The form of the quark $\overline{MS}$ splitting functions and coefficient function that resums all soft singularities in DIS and Drell-Yan is obtained. After obtaining the DIS and Drell-Yan quark coefficient functions that resum leading logarithms and next to leading logarithms, PDF's and higher twist are fitted with and without resummation to the DIS CCFR, BCDMS, SLAC and H1 data sets. These PDF's are then used to determine the effect of resummation on predictions of Tevatron and future LHC Drell-Yan cross sections. Variation of the renormalization and factorization scales is performed to determine if the dependence on these two quantities is reduced in the case that resummation is used, both in the fits to DIS data and in the Drell-Yan predictions.

Independently, an improved treatment of heavy quarks in the calculation of F_2 is investigated. A major simplification of the VFNS is described, and shown theoretically to be as perturbatively good as the VFNS. A fit of PDF's and higher twist to BCDMS and SLAC data is performed with variations of the threshold scales, to determine whether there is less threshold scale dependence in the VFNS than in the ZM-VFNS.

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Chapter 1

Introduction

The Standard Model is the combination of two theories which are mutually compatible, Quantum Chromodynamics (QCD) and electroweak theory, each of which is a quantum field theory (QFT). It is believed to be the most accurate theory known for predicting physics up to collider energies. However, exact analytical solutions to the Standard Model do not exist at present, so either approximate analytical methods or numerical methods must be used to extract phenomenology from it. This limits its range of applicability and verifiability. At present, the main method of testing the Standard Model against nature is by the use of colliders, since the corresponding theoretical predictions of these experimental results are calculable approximately. All phenomenology so far obtained from the Standard Model shows, where applicable, no serious deviation from experimental data.

The most popular analytical method used to solve the Standard Model approximately is perturbation theory. Its full range of applicability is unknown, since the theoretical error in the perturbative calculation of an observable can only ever be obtained at best as an order of magnitude. Yet it has been very successful at giving results that agree with experimental data.

QCD is the part of the Standard Model that is intended to describe hadrons. Two major experimental processes undergone in colliders to test QCD are deep inelastic scattering (DIS), which is the scattering of leptons with hadrons, and

hadron-hadron scattering. The third major one is e^+e^- scattering, but this does not probe hadronic structure, and we will not consider this process in this thesis. In this chapter we outline the basics of perturbation theory as well as hadronic processes.

1.1 Perturbation Theory

After particles in a collider come together at high energy, particles are produced and move away from each other according to the laws of physics that are most apparent at high energy. In a collider, a beam of particles is scattered off a target, or two beams of particles are scattered off one another, and the cross section σ for a given set of final states is measured, where

$$\sigma = \frac{\text{probability}}{\text{flux} \times \text{time}}. \tag{1.1.1}$$

The flux and time are set and known in the experiment, but the probability must be calculated from theory. We consider the Standard Model as a candidate theory. Since the Standard Model is a quantum theory, the probability is given by the modulus squared of the amplitude, which is a matrix element constructed according to the process in question. In certain cases one can use perturbation theory to calculate amplitudes, or certain properties of amplitudes, for scattering processes.

In perturbative QFT, the amplitude for a given scattering process is represented by a sum of *Feynman diagrams* constructed for that process according to a set of rules depending on the specific QFT - each diagram gives a complex number found from a set of *Feynman rules* derived from the QFT, and the sum of the complex numbers gives the amplitude. One first computes the amplitude with the "bare" particles and "bare" coupling g_0 of the QFT. (Note that sometimes more than one coupling is relevant for a process.) It is more convenient to use the *coupling constant* α_0, defined by

$$\alpha_0 = \frac{g_0^2}{4\pi}. \tag{1.1.2}$$

Furthermore, it turns out that diagrams proportional to α_0^n contain an explicit factor π^{-n}, and so it is convenient to use an alternative form of the coupling constant, a_0, defined by

$$a_0 = \frac{\alpha_0}{2\pi}. \tag{1.1.3}$$

Each diagram is proportional to a power of a_0, so the amplitude is a series in a_0. When summing over all intermediate states according to quantum theory, we encounter states of infinite energy. These give rise to divergences in some Feynman diagrams. To deal with these divergences, they must first be *regularized*, for example by imposing an upper energy and momentum limit Λ on all intermediate states. The diagrams, and therefore the coefficients of the series for the amplitude in a_0, are singular only as $\Lambda \to \infty$. We then demand that a_0 is such that the amplitude is finite in this limit. This is done by writing a_0 as an infinite series in a *renormalized coupling* a, where a is numerically well defined, and where the coefficients of this series are singular as $\Lambda \to \infty$ in such a way that the final series for the amplitude in a is non singular in this limit. There is some freedom in the choice of this series since, writing a as a series in a new coupling a', the new series for the amplitude in a' is also non singular as $\Lambda \to \infty$, so a' can also be a renormalized coupling. The choice of the series for a_0 in a is called the *renormalization scheme*. Since the series coefficients for a_0 in a are dimensionless and depend on Λ, these coefficients must also depend on another scale μ_R, the *renormalization scale*. Thus $a = a(\mu_R^2)$. a can be interpreted as the result of absorbing all processes of energy greater than energies around μ_R into a_0. The bare particles and a_0, which are defined at infinite energy, are replaced with "renormalized" particles and $a(\mu_R^2)$ which are defined at the scale μ_R. Physically, a renormalized particle can be interpreted as a bare particle surrounded by a cloud of particles. Experimentalists identify a particle as a system with certain good quantum numbers, and not as a bare point-like particle.

The truncated perturbation series must be convergent, or at least asymptotic,

if it is to approximate the all orders result[1]. Furthermore, each term in the series should be an order of magnitude less than the preceding term if the error on the truncated series is claimed to be one order of magnitude less than the truncated series. To help us achieve this, we use the fact that while the series in a for the amplitude is independent of μ_R and the renormalization scheme to all orders, the truncated series is not. μ_R and the renormalization scheme must be carefully chosen such that the truncated series is as close to the all orders result as possible; if possible they should be chosen such that a is sufficiently small, and such that the series' coefficients are all of a similar order of magnitude (or successively decreasing) to ensure as much as possible that the truncated series is close to the all orders result. Usually, μ_R should be chosen to be of the order of the typical energy scale E of the process, to ensure the ubiquitous $\ln^n \frac{E^2}{\mu_R^2}$ terms in the perturbation series do not spoil the convergence of the series.

We should note at this point that "removing" (or rather reorganizing) large terms, such as the $\ln^n \frac{E^2}{\mu_R^2}$ terms discussed above, in a perturbation series in order to improve the convergence of the series is a *necessary* step in ensuring that the series gives an accurate picture of the all orders result, but it is not *sufficient.* Certainly, it must be ensured that a) the large terms are removed at each and every order and b) that in the process of reorganizing the large terms the all orders result does not change, before a calculation to $O(a^n)$ can be claimed to have an error of $O(a^{n+1})$. (Note this means that a series must be defined to all orders, i.e. such that the calculation of each term is unambiguous, before it can be truncated.) But even then there is still a high level of uncertainty in the error. The problem of the uncertainty due to truncation is reflected in the uncertainty in the choice of μ_R. However, if the truncated series is a good approximation to the infinite series, it should not depend too much on μ_R, provided $\mu_R = O(E)$. If the series is highly dependent on μ_R, what value do we choose for μ_R? However, the value chosen for E, and whether making μ_R close to E really makes a good, even convergent, series, are questions for which it is not possible to provide a sufficient answer to. The best we can do is to fit the perturbative prediction

[1]We use the term "all orders" or "infinite order" to refer to the exact value of the matrix element we are trying to compute, but it must be understand that the infinite series that we are truncating may only be asymptotic *and* divergent.

with different (but not too different) values of μ_R to experimental data. If all the perturbative predictions with the μ_R variations fit the data well, we can assume from consistency that the prediction is a good approximation to the infinite series, and the QFT from which the truncated series was obtained explains the data well. Thus in perturbation theory one does one's best to remove large terms, and then one fits to experimental data as if the removal of the large terms is sufficient. Only if the agreement with experimental data is bad does one question the validity of the truncated series used, and the context that it is used in.

1.2 Introduction To DIS

A lepton is a fermion which only directly interacts through the electroweak force (its colour quantum numbers are null), in contrast to quarks which are fermions with direct interactions through both the electroweak force and QCD. A hadron is defined such that its (observable) quark quantum numbers, each of which is the number of quarks of a given species or *flavour* minus the number of antiquarks of the same flavour, are integers, it is a colour singlet, and it is a bound state of QCD.

From electroweak theory, the Feynman diagram which gives the dominant contribution to the amplitude for

$$e^- + \text{proton} \rightarrow e^- + \text{proton} \tag{1.2.1}$$

is shown in figure 1.1. Time flows from left to right. The scattering is elastic since the initial state particles are the same as the final state particles. Note that here a proton is regarded as fundamental in that all its high energy processes can be predicted from the corresponding Feynman diagrams. Physically, we expect this to be the case when the transferred momentum $|q^2|$ in figure 1.1 is much less than 0.5GeV^2, where the wavelength of the photon is much greater than the proton size of $\sim$ 1fm, and when the errors are sufficiently large enough not to resolve any proton structure, so that the proton can be treated first as a bare

electroweak particle, to give the proton after renormalizing in electroweak theory. For completeness, we note here that in fact QFT's other than electroweak theory, that describe fields which are not elementary fields of the Standard Model but rather arise as a low energy manifestation of the Standard Model, are required to fully describe the proton when its structure is not resolved.

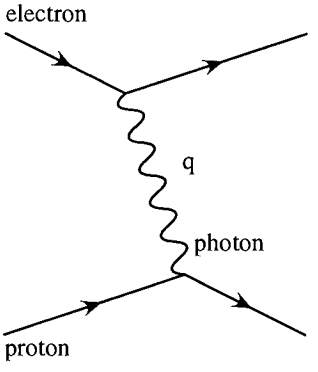

Figure 1.1: Lowest order electron-proton scattering.

As $|q^2|$ moves above 0.5GeV^2 and / or the accuracy of measurement improves, the proton can no longer be treated as fundamental, and the more general inelastic process

$$e^- + \text{proton} \rightarrow e^- + X \tag{1.2.2}$$

is observed, where X is a final state consisting of hadrons. This is DIS [1]. Clearly a new theory is required. For now we assume this theory is a QFT, and proceed ahead. The dominant contribution to the amplitude with respect to electroweak theory for this process is shown in figure 1.2.

We will assume the proton has negligible mass, i.e. $P^2 = 0$. The resulting error from this approximation in the cross section, as we will see in Chapter 2, is of $O\left(\frac{M^2}{Q^2}\right)$, where M is the mass of the proton. The kinematic variables we will use are

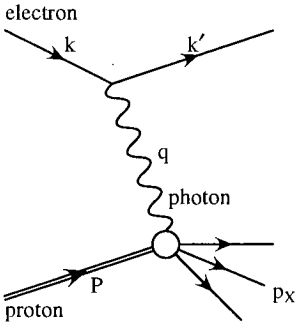

Figure 1.2: Inelastic electron-proton scattering.

$$Q^2 = -q^2 \tag{1.2.3}$$

$$x = \frac{Q^2}{2P \cdot q}. \tag{1.2.4}$$

These variables obey

$$Q^2 \geq 0 \tag{1.2.5}$$

$$0 \leq x \leq 1. \tag{1.2.6}$$

To derive the inequality (1.2.5), we first assume the electron is massless, since its mass is much smaller than the other massive particles we will consider. Now

$$(k' - k)^2 = q^2.$$

But

$$(k' - k)^2 = -2k \cdot k' = -2EE'(1 - \cos\theta),$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$, and E and E' are the energies of the incoming and outgoing electrons respectively, so that $q^2 \leq 0$ and the result follows.

To derive the inequality (1.2.6), we use the result

$$(P+q)^2 = p_X^2. \tag{1.2.7}$$

Expanding and rearranging equation (1.2.7), we obtain

$$x = \frac{1}{1+\frac{p_X^2}{Q^2}}. \tag{1.2.8}$$

Since the final state is a real state, $0 \leq p_X^2 \leq \infty$. So since $Q^2 \geq 0$, the result for x follows.

The amplitude for the process (1.2.2) is found using lowest order electroweak theory to be

$$S_{e^-+p\to e^-+h} = \bar{u}_{\lambda'}(k')ie\gamma_\mu u_\lambda(k)\left(\frac{-ig^{\mu\nu}}{q^2}\right)\langle X(p_X)|iJ_\nu^{em}(0)|\text{proton}(P)\rangle, \tag{1.2.9}$$

where $\bar{u}_{\lambda'}(k')ie\gamma_\mu u_\lambda(k)$ comes from the electron line at the top of figure 1.2, and $\langle X(p_X)|iJ_\mu^{em}(0)|\text{proton}(P)\rangle$ from the proton $\to X$ line. J_μ^{em} is the electromagnetic current operator (with electromagnetic couplings absorbed into it), and depends on the QFT relevant to the proton. Equation (1.2.9) is found by taking the leading order term in electromagnetic couplings of

$$\langle 0|a_{e^-}(k)a_X(p_X)e^{i\int d^4x\mathcal{H}_I(x)}a_{e^-}^+(k')a_{\text{proton}}^+(P)|0\rangle, \tag{1.2.10}$$

where operators are in the interaction picture. The creation and annihilation operators of e^- commute or anticommute with those of the hadrons.

We shall only consider *inclusive processes*, where only the outgoing electron is detected, so that in the theory we must sum the cross section over all final states

X. Furthermore, we shall only consider unpolarized DIS, in which all initial and final spins are averaged and summed over respectively. Our results will be derived without considering spin states. Where necessary, the derivations in the case that spin is taken into account are given in appendix B.1. Then the inclusive differential cross section $d\sigma$ is

$$d\sigma = d^3\mathbf{k}' L^{\mu\nu}(k,k') \frac{1}{Q^4(P\cdot k)} W_{\mu\nu}(P,q). \tag{1.2.11}$$

Here $L^{\mu\nu}(k,k')$ is the leptonic tensor, equal to the square of the electron line as well as other necessary factors. $L^{\mu\nu}$ is easily calculated from electroweak theory.

$W^{\mu\nu}$ contains the hadronic part of the amplitude squared:

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\sum_X \langle \text{proton}(P)|J^{em\dagger}_\mu(0)|X\rangle\langle X|J^{em}_\nu(0)|\text{proton}(P)\rangle \times (2\pi)^4\delta^4(p_X - q - P). \tag{1.2.12}$$

This is the quantity we wish to study. It can be calculated from experimental data via equation (1.2.11). Now by completeness of the theory,

$$\sum_X |X\rangle\langle X| = 1,$$

and since

$$J^{em}_\mu(y) = e^{i\hat{P}.y} J^{em}_\mu(0) e^{-i\hat{P}.y} \qquad (2\pi)^4\delta^4(p) = \int d^4y e^{ip.y},$$

we obtain

$$W_{\mu\nu}(P,q) = \frac{1}{4\pi}\int d^4y e^{iq\cdot y}\langle \text{proton}(P)|J^{em\dagger}_\mu(y)J^{em}_\nu(0)|\text{proton}(P)\rangle. \tag{1.2.13}$$

Note $W_{\mu\nu}$ is Hermitian, i.e. $W_{\mu\nu} = W^*_{\nu\mu}$. The *optical theorem* states that in the inclusive case

$$d\sigma = 2 \text{ Im } d\sigma_f, \tag{1.2.14}$$

where $d\sigma_f$ is the forward scattering amplitude, i.e. the initial and final state of $d\sigma_f$ are both equal to the initial state of $d\sigma$. Since $L_{\mu\nu}$ is the same in both $d\sigma$ and $d\sigma_f$, we have

$$d\sigma = 2\text{Im}\left(d^3\mathbf{k}' L^{\mu\nu}(k,k')\frac{1}{Q^4(P\cdot k)}T_{\mu\nu}(P,q)\right), \tag{1.2.15}$$

where

$$T_{\mu\nu}(P,q) = \frac{i}{4\pi}\int d^4y e^{iq\cdot y}\langle \text{proton}(P)|T\{J^{em}_\mu(y), J^{em}_\nu(0)\}|\text{proton}(P)\rangle. \tag{1.2.16}$$

Since $L_{\mu\nu}$ is real and symmetric, we find that $T_{\mu\nu}$ then has the decomposition

$$2T_{\mu\nu} = H_{\mu\nu} + iW_{\mu\nu}, \tag{1.2.17}$$

where $H_{\mu\nu}$ is Hermitian. Equation (1.2.17) is then just another way of expressing the optical theorem. The properties of $T_{\mu\nu}$ can therefore be used to obtain the properties of $W_{\mu\nu}$. $T_{\mu\nu}$ can be calculated from the diagram in figure 1.3.

Using the Lorentz symmetries of $J^{em}_\mu(y)$ and the Ward identity for the currents, we can write

$$\begin{aligned} W_{\mu\nu}(P,q) = &-\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2) \\ &+\frac{1}{P\cdot q}\left(P_\mu - q_\mu\frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu\frac{P\cdot q}{q^2}\right)F_2(x,Q^2), \end{aligned} \tag{1.2.18}$$

where $F_1(x,Q^2)$ and $F_2(x,Q^2)$ are dimensionless. F_1 and F_2 can be obtained from $W_{\mu\nu}$ from the following projections:

$$F_1 = \left(\frac{2x^2}{Q^2}P^\mu P^\nu - \frac{1}{2}g^{\mu\nu}\right)W_{\mu\nu} \tag{1.2.19}$$

$$F_2 = \left(\frac{12x^3}{Q^2}P^\mu P^\nu - xg^{\mu\nu}\right) W_{\mu\nu}. \tag{1.2.20}$$

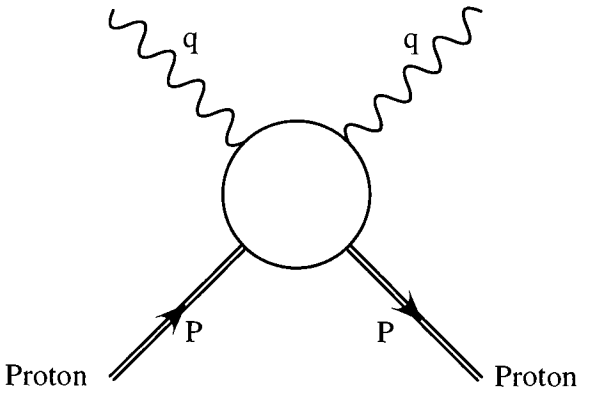

Figure 1.3: Full DIS diagram for $T_{\mu\nu}$.

When the exchanged electroweak particle is a W particle, or if we include Z particle exchange with the photon, equation (1.2.18) is modified to

$$W_{\mu\nu}(P,q) \to W_{\mu\nu}(P,q) + i\epsilon_{\mu\nu\alpha\beta}\frac{P^\alpha q^\beta}{2P\cdot q}F_3(x,Q^2). \tag{1.2.21}$$

Note $W_{\mu\nu}$ remains Hermitian after adding this term. This term violates parity, since the Z and W particles couple to the vector and axial currents, and each current is in a different eigenstate of parity. F_3 can be obtained from $W_{\mu\nu}$ by taking

$$F_3 = \frac{2x}{Q^2}i\epsilon^{\mu\nu\alpha\beta}P_\alpha q_\beta W_{\mu\nu}. \tag{1.2.22}$$

Differential cross sections are the observables measured in colliders. We define $y = \frac{P\cdot q}{P\cdot k}$, which describes the outgoing measured electron. Then in charged current exchange one differential cross section that is measured is

$$\frac{d\sigma^\pm}{dxdy} = \frac{G_F^2 Q^2}{4\pi xy\left(1+\frac{Q^2}{M_W^2}\right)^2}\left[xy^2F_1(x,Q^2) + \left(1-y-\frac{2x^2y^2M^2}{Q^2}\right)F_2(x,Q^2)\right.$$

$$\pm xy\left(1-\frac{1}{2}y\right)F_3(x,Q^2)\Big]. \tag{1.2.23}$$

Thus, keeping x and Q^2 fixed, the $F_{1,2,3}(x,Q^2)$ can then be obtained by fitting them to the above differential cross section measured at at least three different values of y.

Chapter 2

The Parton Model

As explained in section 1.2, the proton does not behave like a fundamental particle. In this chapter we will suppose that the proton is composed of particles called *partons*, which are elementary particles of a QFT. Then, when this QFT is known, one is then able to calculate the cross sections associated with the proton if this QFT is solved here. We will then consider the case that this QFT is QCD, so that the partons that make up the proton are quarks and gluons, the elementary particles of QCD.

The application of perturbation theory to QCD is not as straightforward as the application of perturbation theory to electroweak theory. As in any quantum theory, calculating observables associated with a given process requires summing over intermediate states of all energies. In the application of perturbative QCD to partonic processes, the low energy intermediate states give rise to singularities. Furthermore, the elementary states of QCD are those with quantum numbers of partons, whereas the stable states of experiment are those with hadronic quantum numbers, and it is not clear how to understand hadronic states in terms of partonic states, let alone apply perturbation theory to hadronic states. These two problems are linked because cross sections where one or more of the initial particles is a parton or are partons are not physical cross sections, so we would not expect sensible results for partonic processes.

Fortunately, perturbative QCD is still very useful for extracting phenomenology

from the Standard Model. This is because it is possible to separate the low energy part of an observable from the high energy part. Then the high energy part can be calculated using perturbative QCD. High energy processes, real and virtual, can be understood in terms of partons, but the low energy processes can probably only be understood in terms of hadrons. Certainly there is as yet no complete way of understanding low energy processes in terms of partons. At any rate, the low energy parts can be left as free parameters. This chapter will discuss the formalism and consequences of this separation, and later chapters will draw on the results in this chapter.

2.1 Introduction

In this section we will develop the parton model from a physically intuitive approach, which we will try to justify using QCD later.

The proton contains partons of all possible energy scales[1]. Only the partons at low energy scales make up the bound state. These partons are then *confined.* In other words confinement is a long distance effect. However, we are only observing (or "probing") small distances with the virtual photon, i.e. Q is much larger than the energy scale of confinement, and so we would only observe the almost free partons. Thus we will make the following assumptions about the partons:

1. The partons we can observe with the virtual photon exist in real states, i.e. their spins are physical and their momenta are on-shell, since they are free of interactions. Only interactions can produce virtual states.

2. Since interactions between partons are relevant only at low energy scales, the photon scatters incoherently off the partons within the proton. Thus the total proton cross section is the sum of the partonic cross sections for each parton in each region of phase space, multiplied by the probability of finding that parton in that region of phase space.

[1] "scale" in the context of a parton means the energy at which the physics of the parton becomes relevant, i.e. when its contribution to the observable in question is of the same order of magnitude as the observable.

3. Partons with mass much less than of $O(Q)$ will be treated as massless, since the effects of their masses is over a much larger distance than the distance we are probing. Partons with mass much greater than of $O(Q)$ will be removed from the theory, since they are too heavy to be created, which is anticipated from the fact that if a parton's mass is infinite it cannot be created at all. When Q is of the order of a parton mass, we cannot expect either of the two cases above to be an adequate approximation. This will be remedied in chapter 5.

4. Each parton has a momentum that can be written

$$p = \xi P. \tag{2.1.1}$$

Here we are assuming each parton's momentum is generated solely by the motion of the proton. Any transverse momentum is caused by long distance interactions.

5. The photon scatters off only one high energy scale parton at a time, since the more initial partons the more suppressed the cross section is by powers of $\frac{1}{Q^2}$.

Let the probability of finding a parton of type i and of momentum between ξP and $(\xi + d\xi)P$ in the proton be $f_{0i}(\xi)d\xi$. The $f_{0i}(\xi)$ are called *parton distribution functions* (PDF's). Then assumptions 2 and 5 can be written

$$d\sigma = \sum_i \int_x^1 d\xi d\sigma_i f_{0i}(\xi), \tag{2.1.2}$$

where $d\sigma_i$ is the cross section for parton i, i.e. it is obtained by replacing the proton in $d\sigma$ with parton i. The lower bound x in the integral of equation (2.1.2) follows from the fact that the final state resulting from the photon-parton system must obey $(\xi P + q)^2 \geq 0$, and use of equation (1.2.4). The upper bound 1 follows from the fact that the momentum of what is left of the proton after the parton has left it is $(1-\xi)P$. Since the energy of a real particle is always positive, $P_0 \geq 0$ and $(1-\xi)P_0 \geq 0$, so $\xi \leq 1$.

Then from equations (1.2.11), (2.1.1) and (2.1.2), we find

$$W_{\mu\nu}(P,q) = \sum_i \int_x^1 \frac{d\xi}{\xi} W^i_{\mu\nu}(p,q) f_{0i}(\xi), \tag{2.1.3}$$

where $W^i_{\mu\nu}$ is the parton analogue of equation (1.2.13), given by

$$W^i_{\mu\nu}(p,q) = \frac{1}{4\pi} \int d^4y e^{iq\cdot y} \langle i(p)|J^{em\dagger}_\mu(y) J^{em}_\nu(0)|i(p)\rangle. \tag{2.1.4}$$

Just as in equations (1.2.16) and (1.2.17), the optical theorem for partons states that

$$2T^i_{\mu\nu} = H^i_{\mu\nu} + iW^i_{\mu\nu}, \tag{2.1.5}$$

where

$$T^i_{\mu\nu}(p,q) = \frac{i}{4\pi} \int d^4y e^{iq\cdot y} \langle i(p)|T\{J^{em}_\mu(y), J^{em}_\nu(0)\}|i(p)\rangle. \tag{2.1.6}$$

Once we have found a QFT that describes the partons, equation (2.1.6) can be expressed in terms of, and then calculated from, Feynman diagrams of the class shown in figure 2.1.

For the partons we will use the kinematic variables x' and Q^2, where

$$x' = \frac{Q^2}{2p\cdot q}. \tag{2.1.7}$$

From equations (2.1.1) and (1.2.4),

$$x' = \frac{x}{\xi} \tag{2.1.8}$$

Let $\widehat{F}_{1i}$ and $\widehat{F}_{2i}$ be the F_1 and F_2 cross sections respectively for parton i scattering off a photon, i.e. they are obtained from

$$W^i_{\mu\nu}(p,q) = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) \widehat{F}_{1i}(x', Q^2)$$

$$+\frac{1}{p\cdot q}\left(p_\mu - q_\mu\frac{p\cdot q}{q^2}\right)\left(p_\nu - q_\nu\frac{p\cdot q}{q^2}\right)\widehat{F}_{2i}(x',Q^2). \tag{2.1.9}$$

For Z or W particle exchange, equation (2.1.9) is modified to

$$W^i_{\mu\nu}(p,q) \to W^i_{\mu\nu}(p,q) + +i\epsilon_{\mu\nu\alpha\beta}\frac{p^\alpha q^\beta}{2p\cdot q}\widehat{F}_{3i}(x',Q^2). \tag{2.1.10}$$

We now find the dependence of F_1, F_2 and F_3 on the $\widehat{F}_{1i}$, $\widehat{F}_{2i}$ and $\widehat{F}_{3i}$. From equations (1.2.18), (1.2.19), (2.1.3) and (2.1.9),

$$F_1(x,Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi}\widehat{F}_{1i}(x',Q^2)f_{0i}(\xi). \tag{2.1.11}$$

From equations (1.2.18), (1.2.20), (2.1.3) and (2.1.9),

$$F_2(x,Q^2) = \sum_i \int_x^1 d\xi\widehat{F}_{2i}(x',Q^2)f_{0i}(\xi). \tag{2.1.12}$$

From equations (1.2.21), (1.2.22), (2.1.3) and (2.1.10),

$$F_3(x,Q^2) = \sum_i \int_x^1 \frac{d\xi}{\xi}\widehat{F}_{3i}(x',Q^2)f_{0i}(\xi). \tag{2.1.13}$$

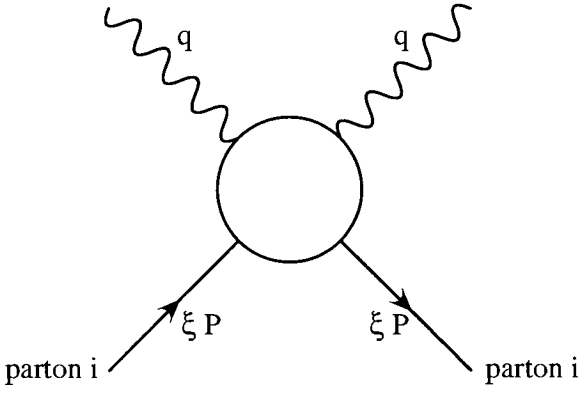

Figure 2.1: Full DIS diagram for $T^i_{\mu\nu}$.

2.2 Mellin Space

Equations (2.1.11) to (2.1.13) are each of the form of a *convolution*,

$$(f \otimes g)(x) = \int_x^1 \frac{d\xi}{\xi} f(\xi) g\left(\frac{x}{\xi}\right). \tag{2.2.1}$$

We will see later that expressions in DIS are generally of this form. We can simplify these types of expressions using the *Mellin transform*. The Mellin transform of a function $f(x)$, hereafter written as $f(N)$, is defined by

$$f(N) = \int_0^1 dx x^{N-1} f(x). \tag{2.2.2}$$

The inverse Mellin transform is given by

$$f(x) = \frac{1}{2\pi i} \int_C dN x^{-N} f(N), \tag{2.2.3}$$

where, as illustrated in figure 2.2, C is a contour that goes from $N = (-\infty, -\infty)$ to $N = (-\infty, \infty)$, and crosses the real axis to the right of all poles in $f(N)$.

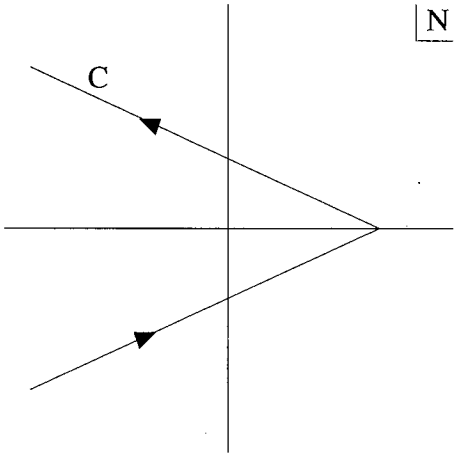

Figure 2.2: The contour for the inverse Mellin transform.

Now, rewriting equation (2.2.1) in the form

$$(f \otimes g)(x) = \int_0^1 dx_1 \int_0^1 dx_2 \delta(x - x_1 x_2) f(x_1) g(x_2), \tag{2.2.4}$$

we find

$$f(N)g(N) = \int_0^1 dx x^{N-1} (f \otimes g)(x), \tag{2.2.5}$$

i.e. the Mellin transform diagonalizes the convolution. Thus equations (2.1.11) to (2.1.13) become

$$F_1(N, Q^2) = \sum_i \widehat{F}_{1i}(N, Q^2) f_{0i}(N) \tag{2.2.6}$$

$$F_2(N-1, Q^2) = \sum_i \widehat{F}_{2i}(N-1, Q^2) f_{0i}(N) \tag{2.2.7}$$

$$F_3(N, Q^2) = \sum_i \widehat{F}_{3i}(N, Q^2) f_{0i}(N). \tag{2.2.8}$$

2.3 The "Naive" Parton Model

In the absence of an interaction with a mass scale between the partons, since F_1, F_2 and F_3 are dimensionless, we would expect that

$$F_{1,2,3}(x, Q^2) = F_{1,2,3}(x). \tag{2.3.1}$$

This is approximately true, and is called *Bjorken scaling*. We will therefore assume the proton is composed only of quarks and antiquarks, described in section 1.2, which only interact through the electroweak force but are otherwise free. We will only consider photon exchange for the time being. Then the cross section for process 1.2.2 is an incoherent sum over the cross sections for quarks q_I and antiquarks $\bar{q}_I$ undergoing the elastic process

$$e^- + q_I(\bar{q}_I) \rightarrow e^- + q_I(\bar{q}_I), \tag{2.3.2}$$

where I labels the quark flavour, and $I = 1, ..., n_f$, where n_f is the number of flavours that contribute to the process. The cross section for this process is obtained from the Feynman diagram in figure 2.3, and we find that, for quarks,

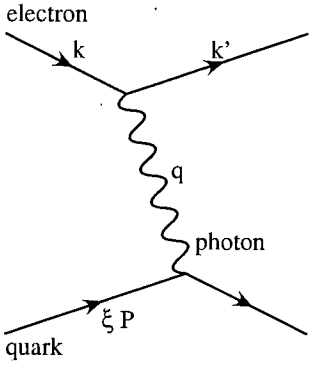

Figure 2.3: Electron-quark scattering.

$$\widehat{F}_{1q_I}(x') = \frac{1}{2} e_{q_I}^2 \delta(1 - x') \tag{2.3.3}$$

$$\widehat{F}_{2q_I}(x') = x' e_{q_I}^2 \delta(1 - x'). \tag{2.3.4}$$

e_{q_I} is the charge of quark q_I. The same results hold for antiquarks. Note $e_{\bar{q}_I} = -e_{q_I}$. Now let $q_{0I}(\xi)$ be the PDF for quarks in the proton, and $\bar{q}_{0I}(\xi)$ that for antiquarks. Thus we obtain

$$F_1(x) = \frac{1}{2} \sum_{I=1}^{n_f} e_{q_I}^2 (q_{0I}(x) + \bar{q}_{0I}(x)) \tag{2.3.5}$$

$$F_2(x) = x \sum_{I=1}^{n_f} e_{q_I}^2 (q_{0I}(x) + \bar{q}_{0I}(x)). \tag{2.3.6}$$

This gives the *Callan-Gross relation*

$$F_2(x) = 2xF_1(x). \tag{2.3.7}$$

Now, the cross section for the absorption of longitudinally polarized photons is given by

$$F_L = F_2 - 2xF_1. \tag{2.3.8}$$

From equation (2.3.7), $F_L = 0$ so there is no absorption of longitudinally polarized photons. This is a direct consequence of the spin-$\frac{1}{2}$ property of the quarks, and the fact that it is experimentally verified to be approximately true provides evidence that the proton is composed of more fundamental spin-$\frac{1}{2}$ constituents.

Formally, the effect of the Z particle must be included for the neutral current process given by equation (1.2.2), as well as for the processes with the electron replaced by a muon, tauon and the antileptons of the latter three particles. This is accounted for by making the replacement in F_1 and F_2 [2, 3]

$$e_{q_I}^2 \to e_{q_I}^2 - 2e_{q_I}V_lV_{q_I}P_Z + (V_l^2 + A_l^2)(V_{q_I}^2 + A_{q_I}^2)P_Z^2, \tag{2.3.9}$$

where

$$V_{e^\pm,\mu^\pm,\tau^\pm} = -\frac{1}{2} + 2\sin^2\theta_W \qquad V_{q_I,\bar{q}_I} = e_{q_I} - \frac{1}{6} - 2e_{q_I}\sin^2\theta_W \tag{2.3.10}$$

$$P_Z = \frac{Q^2}{Q^2 + M_Z^2} \qquad A_{e^\pm,\mu^\pm,\tau^\pm} = -\frac{1}{2} \qquad A_{q_I,\bar{q}_I} = e_{q_I} - \frac{1}{6}. \tag{2.3.11}$$

Therefore, for $Q^2 \ll M_Z^2$, neglecting the effects of the Z particle gives an error of $O\left(\frac{Q^2}{M_Z^2}\right)$. F_3 for the process of equation (1.2.2) is given by

$$F_3(x) = \sum_{I=1}^{n_f}(-2e_{q_I}A_lA_{q_I}P_Z + 4V_lA_lV_{q_i}A_{q_I}P_Z^2)(q_{0I}(x) - \bar{q}_{0I}(x)). \tag{2.3.12}$$

For the charged current process

$$\nu_l + \text{proton} \rightarrow l + X, \tag{2.3.13}$$

which is an incoherent sum over processes of the form

$$\nu_l + q_{I'}(\bar{q}_{J''}) \rightarrow l + \bar{q}_{J''}(q_{I'}), \tag{2.3.14}$$

where I' is odd and J'' is even, we have

$$F_1 = \frac{1}{2} \sum_{I' \leq n_f} \sum_{J'' \leq n_f} |V_{J''I'}|^2 (q_{0I'}(x) + \bar{q}_{0J''}(x)) \tag{2.3.15}$$

$$F_2 = x \sum_{I' \leq n_f} \sum_{J'' \leq n_f} |V_{J''I'}|^2 (q_{0I'}(x) + \bar{q}_{0J''}(x)) \tag{2.3.16}$$

$$F_3 = \sum_{I' \leq n_f} \sum_{J'' \leq n_f} |V_{J''I'}|^2 (q_{0I'}(x) - \bar{q}_{0J''}(x)). \tag{2.3.17}$$

$V_{J''I'}$ is the *CKM matrix*, that describes $W^{\pm}$ emission or absorption by a quark in a flavour eigenstate and the consequent transition to other flavours. For the charged current process where the initial lepton is an antineutrino, the quarks and antiquarks in equations (2.3.15) to (2.3.17) must be interchanged.

2.4 Perturbative QCD

Since the sum of the momenta of all the quarks must equal the momentum of the proton, we expect that

$$\sum_{I=1}^{n_f} \int_0^1 d\xi \xi \left(q_{0I}(\xi) + \bar{q}_{0I}(\xi)\right) = 1. \tag{2.4.1}$$

From experiment, the right hand side is ~ 0.5. This suggests that there are other particles present in the proton. Furthermore, $F_{1,2,3}$ exhibit *scaling violation*, in which the Bjorken scaling of equation (2.3.1) is violated, so $F_{1,2,3}$ become dependent on Q^2. This suggests quarks have an interaction with a mass scale.

The elementary particles of QCD are quarks and gluons, and we shall now consider QCD as a candidate for predicting hadron phenomenology and assume the missing particles are gluons. In this section we outline the parts of perturbative QCD [4] that will be relevant later. Consider the β *function*,

$$\beta(a_s(\mu_R^2)) = \frac{da_s(\mu_R^2)}{d\ln\mu_R^2}, \tag{2.4.2}$$

where

$$a_s = \frac{\alpha_s}{2\pi}, \tag{2.4.3}$$

where α_s is the *running coupling*. We expand $\beta(a_s)$ as

$$\beta(a_s) = -\sum_{n=0}^{\infty} \beta_n a_s^{n+2}, \tag{2.4.4}$$

where, from [5], we have explicitly used the fact that the $O(1)$ and $O(a_s)$ terms are zero. From [5], for n_f massless flavours and in the $\overline{MS}$ scheme,

$$\beta_0 = \frac{11}{6}C_A - \frac{2}{3}T_R n_f \tag{2.4.5}$$

$$\beta_1 = \frac{17}{6}C_A^2 - C_F T_R n_f - \frac{5}{3}C_A T_R n_f \tag{2.4.6}$$

$$\begin{aligned}\beta_2 = \frac{2857}{432}C_A^3 + \frac{1}{4}C_F^2 T_R n_f - \frac{205}{72}C_F C_A T_R n_f - \frac{1415}{216}C_A^2 T_R n_f \\ + \frac{11}{18}C_F T_R^2 n_f^2 + \frac{79}{108}C_A T_R^2 n_f^2,\end{aligned} \tag{2.4.7}$$

where

$$C_A = 3 \qquad C_F = \frac{4}{3} \qquad T_R = \frac{1}{2}. \tag{2.4.8}$$

β_0 is positive for $n_f \leq 16$. Thus, with this condition on n_f, from equation (2.4.2), the coupling $a_s(\mu_R^2)$ decreases as μ_R increases. Solving equation (2.4.2) gives

$$a_s(\mu_R^2) = \frac{1}{\beta_0 \ln(\mu_R^2/\Lambda_{QCD}^2)}\left(1 - \frac{\beta_1}{\beta_0^2}\frac{\ln\ln(\mu_R^2/\Lambda_{QCD}^2)}{\ln(\mu_R^2/\Lambda_{QCD}^2)}\right.$$
$$\left. + \left(\frac{\beta_1}{\beta_0^2}\right)^2 \left(\frac{\ln^2 \ln \mu_R^2/\Lambda_{QCD}^2 - \ln\ln\mu_R^2/\Lambda_{QCD}^2 + \frac{\beta_0\beta_2}{\beta_1^2} - 1}{\ln^2(\mu_R^2/\Lambda_{QCD}^2)}\right)\right)$$
$$+ O\left(\frac{1}{\ln^4(\mu_R^2/\Lambda_{QCD}^2)}\right). \tag{2.4.9}$$

This is the *next-next-to-leading-order* (NNLO) form of $a_s(\mu_R^2)$. Λ_{QCD} is the integration constant, and is a fundamental parameter of QCD. It depends on the renormalization scheme and n_f. An alternative solution is

$$a_s(s\mu_R^2) = \frac{a_s(\mu_R^2)}{1 + a_s(\mu_R^2)\beta_0 \ln s}$$
$$- \frac{\beta_1}{\beta_0} a_s^2(\mu_R^2) \frac{\ln\left(1 + a_s(\mu_R^2)\beta_0 \ln s\right)}{\left(1 + a_s(\mu_R^2)\beta_0 \ln s\right)^2} + O(a_s^3(a_s \ln s)^n). \tag{2.4.10}$$

The series in a_s for the physical quantity one is trying to compute from perturbative QCD will contain $\ln^n \frac{Q^2}{\mu_R^2}$ terms, which are large when Q and μ_R are very different. Thus μ_R must be set to the order of the typical energy scale Q of the process for perturbative QCD to work. Consequently perturbative QCD will only be applicable for processes of sufficiently high energy, where $\mu_R \sim Q \gg \Lambda_{QCD}$, since here a_s is small.

Thus we see that in the limit that the energy scale of a process goes to infinity, only the $O(a_s^0)$ term of the perturbation series contributes to the observable in question. Physically the strong force, measured by a_s, appears to vanish, i.e. quarks appear free, as the energy scale of a process goes to infinity. This is called

asymptotic freedom. The "naive" parton model is then the $O(a_s^0)$ part of the parton model with QCD.

At energy scales close to Λ_{QCD}, perturbation theory does not work, i.e. we cannot treat partons as free particles that have been perturbed by a small interaction, since here a_s as defined by equation (2.4.9) cannot be used as an expansion parameter. Thus confined partons have an energy scale of $O(\Lambda_{QCD})$ and less.

2.5 The Parton Model From Full QCD

To test a theory of nature and discover new physics beyond this theory, the kinematic range and number of types of experiment that can be both performed experimentally and predicted theoretically needs to be as large as possible. Furthermore, it is always important to know not just the experimental errors of the observables measured but also to have some idea of the theoretical error in the calculation. To calculate and observe contributions to hadronic physics not directly part of QCD (such as Higgs physics or supersymmetry) requires knowledge of the QCD background, so we need to be able to calculate the QCD contributions to diagrams like that in figure 1.3, and have some idea of the error on our calculations. In the next few sections we will outline how this can be achieved.

Assumptions 1 to 5 of section 2.1 in the case of inclusive DIS will now be given some justification using QCD. For PDF's, we will use the notation f_{0i}, $i = 0, ..., n_f$, where

$$f_{0I} = q_{0I} \qquad f_{0\bar{I}} = \bar{q}_{0I}, \qquad I = 1, ..., n_f, \tag{2.5.1}$$

and f_{00} is the PDF of the gluon. However, we shall always write f_{00} explicitly as g_0. Note the conventions we are using: Uncapitalized indices, e.g. i, can take the values $i = 1, ..., n_f$, $i = \bar{1}, ..., \bar{n}_f$ and $i = 0$. The symbol $\sum_i$ will imply a sum over all these possible values of i. Capitalized indices, e.g. I, can take on the values $I = 1, ..., n_f$ and $I = \bar{1}, ..., \bar{n}_f$. The symbol $\sum_I$ will imply a sum over all these possible values of I.

A heuristic physical argument for assumptions 1, 2 and 4, from [6], is as follows: A parton at a scale of $O(\Lambda_{QCD})$, which does not fall into any of the types of partons categorized in assumptions 1, 2 and 4, has a cross sectional area of $O(1/\Lambda_{QCD}^2)$, and we are probing with a photon of momentum of $O(Q)$ so that the interaction area is of $O(1/Q^2)$. The probability of observing this parton is the area we are probing divided by the area of the parton, which is of $O\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$.

If we expand the full cross section in powers of $\frac{\Lambda_{QCD}^2}{Q^2}$, then each term is referred to as having a given *twist*. The first term in this expansion is called the *leading region* or *leading twist*, which is what we shall be interested in.

The error in neglecting a parton mass $m \ll Q$ from power counting is of $O\left(\frac{m^2}{Q^2}\right)$. In other words we do a Taylor expansion of the cross section in $\frac{m^2}{Q^2}$, assuming the cross section is well defined in the massless limit. The error in neglecting a parton of mass $m \gg Q$ altogether is of $O\left(\frac{Q^2}{m^2}\right)$, assuming the cross section is well defined in the infinite mass limit. This latter assumption is called the *decoupling theorem* [7]. This provides some justification for assumption 3.

We now give more complete justifications for the parton model and assumptions 1, 2 and 4, from a more detailed study of QCD. We expand the diagram in figure 1.3 as in figure 2.4, where the diagrams $D_{\mu\nu}$, $C_{0\mu\nu}$, K_0 and T_0 are 2 particle irreducible (2PI) in the t-channel, the lines joining them can either be quarks or gluons, and each diagram contains the upper propagators (so each diagram is not strictly 2PI, but only in these propagators). Note that we are considering $W_{\mu\nu}$, so each diagram has a cut, given by the dashed vertical line. The final states, which are summed over, sit on the cut. This expansion can be derived more formally using effective fields. Note the joining lines are part of loops so they have sums and integrations over relevant quantum numbers.

We define

$$F_\Theta = \Theta^{\mu\nu} W_{\mu\nu}, \tag{2.5.2}$$

where F_Θ is essentially a structure function such as any one of the $F_{1,2,3}$, and $\Theta^{\mu\nu}$ is some real symmetric tensor of our choosing, for example one of those in

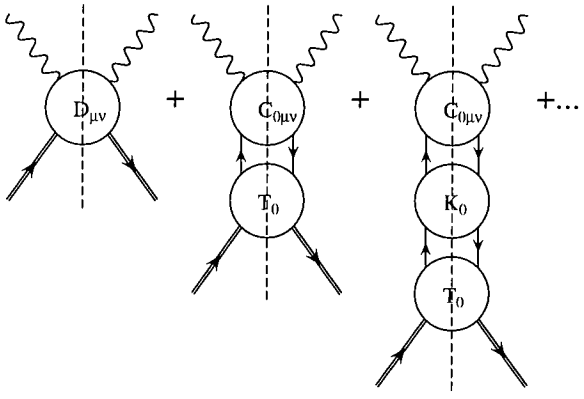

Figure 2.4: Decomposition Of $W_{\mu\nu}$.

equations (1.2.19), (1.2.20) and (1.2.22). Then, writing $C_0^\Theta = \Theta^{\mu\nu} C_{0\mu\nu}$, figure 2.4 can be written

$$F_\Theta = C_0^\Theta \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot T_0 + D^\Theta, \tag{2.5.3}$$

where

$$\frac{\mathbb{1}}{\mathbb{1} - K_0} = \mathbb{1} + K_0 + K_0 \cdot K_0 + ..., \tag{2.5.4}$$

and where the "$\cdot$" refers to the integrations and sums in loops. In other words, omitting parton indices for brevity, for any two 2PI diagrams A and B,

$$A \cdot B = \sum_s \int d^4p A^s(q,p) B^s(p,P). \tag{2.5.5}$$

s refers to the spin combination of the two joining lines between A and B. Note A also has a spin index for its two upper lines, and B also has a spin index for its two lower lines, but we have suppressed these indices for brevity.

For massless particles, the integral in equation (2.5.5) is singular, which arises from the $p^2 \to 0$ region of the integral between the two 2PI diagrams. These singularities are called *mass singularities*. We shall discuss how to reshuffle these singularities later. For now we assume all partons have a mass unless otherwise

stated, to regulate these singularities. It has been proven in [8] that the diagrams $C_0^{\ominus}$, K_0, T_0 and $D^{\ominus}$ are free of UV and mass singularities in a physical light-like gauge (e.g. when $A^+ = 0$) when written as an expansion in the renormalized coupling.

Let us define a projection operator $\hat{O} = \hat{O} \cdot \hat{O}$ that acts between A and B in the form

$$A \cdot \hat{O} \cdot B = \sum_s \int d^4p \sum_t \int d^4p' A^s(q,p') \hat{O}^{st}(p',p) B^t(p,P). \tag{2.5.6}$$

$\hat{O}$ is diagonal in parton indices, which we have chosen to omit for brevity. We shall choose $\hat{O}$ such that it sets the momentum of the lower incoming parton lines in A to that given by equation (2.1.1), so that ξ is the only variable left to integrate over in the "$\cdot$", and sets the polarization of these parton lines to be physical. To make this more explicit, we use the *Sudakov parameterization* for p,

$$p = \beta_p P + p_T + \left(\frac{p^2 - p_T^2}{2\beta_p P \cdot \bar{P}} \right) \bar{P}, \tag{2.5.7}$$

where

$$P^2 = \bar{P}^2 = P \cdot p_T = \bar{P} \cdot p_T = 0 \qquad P \cdot \bar{P} \neq 0. \tag{2.5.8}$$

For example, writing 4-vectors as $V^\mu = (V^+, V^-, \mathbf{V}_T)$, where $V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^3)$ and $\mathbf{V}_T = (V_T^1, V_T^2)$, then in the *collinear frame*,

$$P = (P^+, 0, \mathbf{0}) \qquad \bar{P} = (0, P^+, \mathbf{0}) \qquad p_T = (0, 0, \mathbf{p_T}), \tag{2.5.9}$$

so that

$$p = \left(\beta_p P^+, \frac{p^2 - p_T^2}{2\beta_p P^+}, \mathbf{p_T} \right). \tag{2.5.10}$$

The form of $\hat{O}$ we will choose, temporarily restoring parton indices i and j, is then

$$\hat{O}_{ij}^{rt}(p',p) = \begin{cases} \delta^{(4)}(p' - \beta_p P) & \text{if } r = t = \text{physical and } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.11}$$

Note that

$$\delta^{(4)}(p' - \beta_p P) = 2\beta_{p'}\delta(\beta_{p'} - \beta_p)\delta(p'^2)\delta^{(2)}(\mathbf{p_T'}). \tag{2.5.12}$$

Now, using

$$d^4p = \frac{d\beta_p}{2\beta_p} dp^2 d^2\mathbf{p_T}, \tag{2.5.13}$$

the action of $\hat{O}$ between any two diagrams A and B is

$$A \cdot \hat{O} \cdot B = \sum_{s=\text{phys}} \int \frac{d\beta_p}{2\beta_p} \int_0^\infty dp^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{p_T} A^s(q, \beta_p P) B^s(p, P). \tag{2.5.14}$$

Note that $\hat{O}$ reduces the "$\cdot$" between two diagrams, which involves sums and integrations over 4-momenta, to a convolution over a single variable and a sum over flavours and spin. We now find the order of magnitude of the difference between $A \cdot B$ and $A \cdot \hat{O} \cdot B$. Our arguments will be mainly heuristic, and are given in more detail in [9]. Let us consider the case that $|q^2| \gg P^2$, and $P^2 = O(\Lambda_{QCD}^2)$. The integrand of the $\mathbf{p_T}$ integrals and the p^2 integral in equation (2.5.14) reads

$$\sum_{s=\text{phys}} \int \frac{d\beta_p}{2\beta_p} A^s(q, \beta_p P) B^s(p, P), \tag{2.5.15}$$

whereas that in equation (2.5.5) reads

$$\sum_s \int \frac{d\beta_p}{2\beta_p} A^s(q, p) B^s(p, P). \tag{2.5.16}$$

The largest spin components are the physical ones, since these components of B contain the largest momenta in B. When $\mathbf{p_T}^2$ and p^2 are small compared with Q^2, then from power counting (i.e. Taylor's Theorem), only terms of $O\left(\frac{p^2}{Q^2}\right)$ and $O\left(\frac{p_T^2}{Q^2}\right)$ in $A^r(q,p)$ are changed when the $\hat{O}$ operator is inserted, so the $\hat{O}$ operator only causes a relative difference of $O\left(\frac{p^2}{Q^2}\right)$ and $O\left(\frac{p_T^2}{Q^2}\right)$ to $A \cdot B$, prior to integration over $\mathbf{p_T}$ and p^2. However, when $\mathbf{p_T}^2$ and p^2 are of $O(Q^2)$ or larger, all that power counting can tell us is that

$$A \cdot \hat{O} \cdot B = A \cdot B \left(1 + O\left(\left(\frac{\text{highest virtuality in } B}{\text{lowest virtuality in } A}\right)^2\right)\right). \tag{2.5.17}$$

The lowest virtuality in A is about the same as the highest virtuality in B, so $A \cdot \hat{O} \cdot B$ does not give an adequate approximation to $A \cdot B$.

We now use the $\hat{O}$ operator to write F_Θ in the form of equation (2.1.3) in such a way that the error is no more than of $O\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$, which we can do as follows [9]. Let us define a quantity

$$r_\Theta = C_0^\Theta \cdot (\mathbb{1} - \hat{O}) \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0 \cdot (\mathbb{1} - \hat{O})} \cdot T_0 + D^\Theta. \tag{2.5.18}$$

The first term in equation (2.5.18) is like the first term of equation (2.5.3), but with the operator $\mathbb{1} - \hat{O}$ placed wherever 2 2PI diagrams are joined. r_Θ contains no UV singularities, or rather all UV singularities cancel in r_Θ since the $\mathbb{1} - \hat{O}$ operator suppresses all UV divergences occurring in graphs above it. The $\mathbb{1} - \hat{O}$ operator also removes the mass singularities that arise between the 2PI diagrams, since then the integrand of the p^2 integral is zero as $p^2 \to 0$. The $\mathbb{1} - \hat{O}$ operator causes a strong decrease in virtuality on going from the 2PI diagram above the operator to the one below. Within a given 2PI diagram, the highest virtuality is about the same as the lowest virtuality, or more precisely when the range of virtualities in a 2PI diagram is large the diagram is power suppressed. (Although the larger the diagram (i.e. the higher the order in a_s), the greater the range of virtualities can be without this suppression.) Thus, since the lowest virtuality in C_0^Θ is of $O(Q^2)$ and the highest virtuality in T_0 is of $O(\Lambda_{QCD}^2)$, we find from

equation (2.5.17) that the first term in equation (2.5.18) is of $O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$ (taking F_Θ to be of $O(1)$). Finally, it has been proven in [10] that all contributions to the leading region come only from diagrams that are two particle reducible, so that D^Θ is of $O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$. Thus

$$r_\Theta = O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right). \tag{2.5.19}$$

Now, we can use equation (2.5.18) to rewrite equation (2.5.3)

$$F_\Theta = C_0^\Theta \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot \hat{O} \cdot \hat{O} \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0 \cdot (\mathbb{1} - \hat{O})} \cdot T_0 + r_\Theta. \tag{2.5.20}$$

By definition of a bare partonic structure function,

$$\hat{F}_\Theta = C_0^\Theta \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot \hat{O}, \tag{2.5.21}$$

since the right hand side of equation (2.5.21) is just the imaginary part of the diagram in figure 2.1. Thus from equations (2.5.20) and (2.5.21), we can make contact with the parton model in the form of equation (2.1.3) if we take

$$f_0 = \hat{O} \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0 \cdot (\mathbb{1} - \hat{O})} \cdot T_0. \tag{2.5.22}$$

Equation (2.5.22) defines the "bare" PDF's in QCD, and is in principle calculable (although it contains mass singularities, which we shall deal with later). Here "in principle" means that no sufficient technique has yet been fully formulated, but work is in progress [11, 12].

Furthermore, our expression for F_Θ in equation (2.5.20) contains all QCD contributions, where the first term is the leading twist term, and r_Θ contains all higher twist terms, and note that r_Θ is defined by equation (2.5.18) and so it too is in principle calculable.

Therefore, in summary, equation (2.1.3) in massless QCD is correct up to terms of $O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$.

2.6 The Parton Model With QCD

We now use perturbative QCD to calculate the $\widehat{F}_{1,2,3i}$ [2, 13]. This will give us some information on $W_{\mu\nu}$, or equivalently $F_{1,2,3}$. The results of section (2.3) are the zeroth order (tree level) results of perturbative QCD. To be definite we will only consider the F_2 calculation, but the procedure is essentially the same for $F_{1,3}$. We take all partons to be massless.

We first consider the process

$$\gamma + q_I \rightarrow \gamma + q_I. \tag{2.6.1}$$

There are four $O(a_s)$ diagrams. One contains a mass singularity, and is shown in figure 2.5.

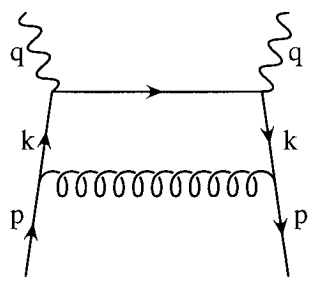

Figure 2.5: Diagram containing the only mass singular part of $O(a_s)$ photon-quark scattering.

Applying the Feynman rules to this diagram, performing all integrals except the k^2 integral and then taking the imaginary part gives a contribution to $\widehat{F}_{2q_I}$ of

$$\widehat{F}_{2,Fig\ 2.5} = e_{q_I}^2 a_s(\mu_R^2) x' P_{qq}^{(0)}(x') \int_0^{2p\cdot q} \frac{dk^2}{k^2}. \tag{2.6.2}$$

The function $a_s(\mu_R^2)P_{qq}^{(0)}(x')$ is the leading order part of the number density of quarks of momentum $x'p$ in a quark of momentum p. We will discuss this function in more detail later. Equation (2.6.2) contains a mass singularity, which arises from the $k^2 \to 0$ part of the integral. Alternatively one could perform the k^2 integral before the k_T integral, and then we would see the mass singularity arising as $k_T \to 0$, i.e. when the gluon is collinear to the quark. This initial state singularity corresponds to a long distance or *soft* part of the strong interaction which is not calculable in perturbation theory. The *KLN theorem* [14, 15] states that initial state infrared singularities will only cancel when the initial states are physical. These physical initial states can be summations over unphysical states, and in the case of DIS, physical initial states are arrived at by convoluting the bare partonic cross sections $\widehat{F}_{2i}$ with bare PDF's and summing over all partons, since then the initial state is hadronic, and therefore physical. Giving all partons a mass would remove this mass singularity. For now we regularize the mass singularity in equation (2.6.2) by replacing the "0" in the lower bound of the integral with a small quantity κ^2, to get

$$\widehat{F}_{2,Fig2.5} = e_{q_I}^2 a(\mu_R^2) x' P_{qq}^{(0)}(x') \ln\left(\frac{2p\cdot q}{\kappa^2}\right). \tag{2.6.3}$$

The limit $\kappa^2 \to 0$ will be understood later. There are 3 other (non singular) $O(a_s)$ one loop diagrams, each with one gluon propagator, to be included with figure 2.5. With the tree level result in equation (2.3.4) we find the total F_2 cross section to $O(a_s)$ for process (2.6.1) to be

$$\widehat{F}_{2q_I}(x', Q^2) = x' e_{q_I}^2 \left(\delta(1-x') + a_s(\mu_R^2)\left(P_{qq}^{(0)}(x') \ln\frac{Q^2}{\kappa^2} + C_{2qNS}^{(1)}(x')\right)\right). \tag{2.6.4}$$

$x' e_{q_I}^2 a_s(\mu_R^2) C_{2qNS}^{(1)}(x')$ is an $O(a_s)$ *hard* part of $\widehat{F}_{2q_I}(x', Q^2)$, and is non singular. The same result holds for antiquarks. According to equation (2.1.12), to obtain the part of the F_2 cross section due to quarks in the proton, which we will denote

as F_{2q}, we must sum over quarks with their appropriate weight, the quark PDF's $q_{0I}(\xi)$ and $\bar{q}_{0I}(\xi)$, to obtain

$$F_{2q}(x,Q^2) = x\sum_{I=1}^{n_f} e_{q_I}^2 \int_x^1 \frac{d\xi}{\xi}(q_{0I}(\xi)+\bar{q}_{0I}(\xi))$$

$$\times\left(\delta(1-x') + a_s(\mu_R^2)\left(P_{qq}^{(0)}(x')\ln\frac{Q^2}{\kappa^2} + C_{2qNS}^{(1)}(x')\right)\right). \tag{2.6.5}$$

Next we do the analogous calculation for the process

$$\gamma + g \to \gamma + g, \tag{2.6.6}$$

which will give us $\widehat{F}_{2g}$. One of the diagrams contributing to $\widehat{F}_{2g}$ is shown in figure 2.6. There is also a second diagram in which the fermion arrows are reversed. There is also a third and fourth diagram where the quark lines are crossed. The total F_2 cross section of this process to $O(a_s)$ is

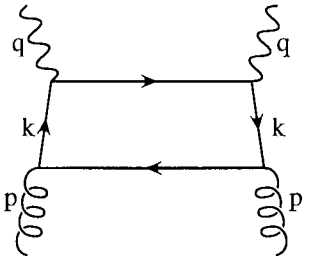

Figure 2.6: One of the $O(a_s)$ photon-gluon fusion diagrams.

$$\widehat{F}_{2g}(x',Q^2) = x'\sum_{I=1}^{n_f} e_{q_I}^2 a_s(\mu_R^2)\left(P_{qg}^{(0)}(x')\ln\frac{Q^2}{\kappa^2} + C_{2g}^{(1)}(x')\right). \tag{2.6.7}$$

Using the gluon PDF, $g_0(\xi)$, the contribution to the total cross section from gluons is

$$F_{2g}(x,Q^2) = x\sum_{I=1}^{n_f} e_{q_I}^2 a_s(\mu_R^2)\int_x^1 \frac{d\xi}{\xi} g_0(\xi)\left(P_{qg}^{(0)}(x')\ln\frac{Q^2}{\kappa^2} + C_{2g}^{(1)}(x')\right). \quad (2.6.8)$$

Note that, to all orders, $\widehat{F}_{2q_I}$, $\widehat{F}_{2\bar{q}_I}$ and $\widehat{F}_{2g}$ do not depend on μ_R as they are cross sections. The independence is accounted for by $\ln^n \frac{Q^2}{\mu_R^2}$ terms appearing at higher orders.

In Mellin space, equation (2.6.5) becomes

$$F_{2q}(N-1,Q^2) = \sum_{I=1}^{n_f} e_{q_I}^2 (q_{0I}(N) + \bar{q}_{0I}(N))$$
$$\times\left(1 + a_s(\mu_R^2)\left(P_{qq}^{(0)}(N)\ln\frac{Q^2}{\kappa^2} + C_{2qNS}^{(1)}(N)\right)\right), \quad (2.6.9)$$

and equation (2.6.8) becomes

$$F_{2g}(N-1,Q^2) = \sum_{I=1}^{n_f} e_{q_I}^2 g_0(N) a_s(\mu_R^2)\left(P_{qg}^{(0)}(N)\ln\frac{Q^2}{\kappa^2} + C_{2g}^{(1)}(N)\right). \quad (2.6.10)$$

We shall be writing our results in Mellin space from now on.

2.7 Factorization

To ensure that the total cross section

$$F_2 = F_{2q} + F_{2g} \quad (2.7.1)$$

is finite, we have to specify the dependence that the "bare" PDF's, $g_0(N)$, $q_{0I}(N)$ and $\bar{q}_{0I}(N)$, must have on the mass singularities such that the cross section is finite, despite our ignorance of the bare PDF's. Our ignorance of them is put into finite *factorized* PDF's $g(N,\mu_F^2)$, $q_I(N,\mu_F^2)$ and $\bar{q}_I(N,\mu_F^2)$, where μ_F is the

factorization scale. Physically, processes of energy less than μ_F are absorbed into the bare PDF's. Explicitly, if we write

$$q_{0I}(N) + \bar{q}_{0I}(N) = \left(1 - a_s(\mu_R^2)P_{qq}^{(0)}(N)\ln\frac{\mu_F^2}{\kappa^2}\right)(q_I(N,\mu_F^2) + \bar{q}_I(N,\mu_F^2))$$

$$-a_s(\mu_R^2)P_{qg}^{(0)}(N)\ln\frac{\mu_F^2}{\kappa^2}g(N,\mu_F^2) + O(a_s^2) \tag{2.7.2}$$

$$g_0(N) = \left(1 - a_s(\mu_R^2)P_{gg}^{(0)}(N)\ln\frac{\mu_F^2}{\kappa^2}\right)g(N,\mu_F^2)$$

$$-\sum_{I=1}^{n_f} a_s(\mu_R^2)P_{gq}^{(0)}(N)\ln\frac{\mu_F^2}{\kappa^2}(q_I(N,\mu_F^2) + \bar{q}_I(N,\mu_F^2)) + O(a_s^2), \tag{2.7.3}$$

then from equation (2.6.9) and (2.6.10), keeping only terms to $O(a_s)$, we find

$$F_2(N-1,Q^2) = \sum_{I=1}^{n_f} e_{q_I}^2\Big[\left(1 + a_s(\mu_R^2)\left(P_{qq}^{(0)}(N)\ln\frac{Q^2}{\mu_F^2} + C_{2qNS}^{(1)}(N)\right)\right)$$

$$\times(q_I(N,\mu_F^2) + \bar{q}_I(N,\mu_F^2))$$

$$+a_s(\mu_R^2)\left(P_{gq}^{(0)}(N)\ln\frac{Q^2}{\mu_F^2} + C_{2g}^{(1)}(N)\right)g(N,\mu_F^2)\Big] + O(a_s^2). \tag{2.7.4}$$

To $O(a_s)$, the absence of mass singularities in F_2 is now explicit. Since $g_0(N)$, $q_{0I}(N)$ and $\bar{q}_{0I}(N)$ are independent of μ_F^2, we can obtain the μ_F^2 dependence of $g(N,\mu_F^2)$, $q_I(N,\mu_F^2)$ and $\bar{q}_I(N,\mu_F^2)$ by differentiating equations (2.7.2) and (2.7.3) with respect to μ_F^2, to find

$$\frac{dg(N,\ln\mu_F^2)}{d\ln\mu_F^2} = a_s(\mu_R^2)P_{gg}^{(0)}(N)g(N,\ln\mu_F^2) + a_s(\mu_R^2)P_{gq}^{(0)}(N)\Sigma(N,\ln\mu_F^2) + O(a_s^2) \tag{2.7.5}$$

$$\frac{d\Sigma(N,\ln\mu_F^2)}{d\ln\mu_F^2} = a_s(\mu_R^2)P_{qq}^{(0)}(N)\Sigma(N,\ln\mu_F^2) + a_s(\mu_R^2)2n_f P_{qg}^{(0)}(N)g(N,\mu_F^2) + O(a_s^2) \tag{2.7.6}$$

$$\frac{dT_{I,J}(N,\mu_F^2)}{d\ln\mu_F^2} = a_s(\mu_R^2)P_{qq}^{(0)}(N)T_{I,J}(N,\mu_F^2) + O(a_s^2), \tag{2.7.7}$$

where

$$\Sigma = \sum_{I=1}^{n_f}(q_I + \bar{q}_I) \qquad T_{I,J} = q_I + \bar{q}_I - q_J - \bar{q}_J. \tag{2.7.8}$$

Therefore, to $O(a_s)$, the dependence of $g(N,\mu_F^2)$, $q_I(N,\mu_F^2)$ and $\bar{q}_I(N,\mu_F^2)$ on μ_F^2 is non-singular.

Note that $F_2(N-1,Q^2)$ is independent of μ_F and μ_R to all orders, but this is not true for truncated results like equation (2.7.4). Note that only the leading order term in equation (2.7.3) was needed in equation (2.7.4), in contrast with equations (2.7.5) and (2.7.6).

It must be noted that the bare quark PDF's are process independent - we would use these same PDF's in, for example, proton-proton collision calculations. Thus by equations (2.7.2) and (2.7.3) the factorized PDF's are the same, when factorized the same way, for any process.

Setting $\mu_F = \mu_R = Q$ in equation (2.7.4) gives the simpler form

$$F_2(N-1,Q^2) = \sum_{I=1}^{n_f} e_{q_I}^2\Big[\Big(1 + a_s(Q^2)C_{2qNS}^{(1)}(N)\Big)(q_I(N,Q^2) + \bar{q}_I(N,Q^2)) + a_s(Q^2)C_{2g}^{(1)}(N)g(N,Q^2)\Big]. \tag{2.7.9}$$

We are free to absorb finite quantities into the factorized PDF's. How we do this is referred to as choosing a *scheme*. For example, one choice of scheme is the DIS scheme in which everything is absorbed into the definition of the $q_I(N,Q^2)$ and $\bar{q}_I(N,Q^2)$, so that

$$F_2(N-1,Q^2) = \sum_{I=1}^{n_f} e_{q_I}^2(q_I(N,Q^2) + \bar{q}_I(N,Q^2)). \tag{2.7.10}$$

2.8 The Factorization Theorem

At higher orders we expect more mass singularities to arise in the Feynman diagrams. To determine whether perturbative QCD has some predictive power in hadron phenomenology, it must be proven to all orders that it is possible to rewrite F_Θ in terms of quantities of which at least some are approximately numerically determinable from perturbative QCD. The bare PDF's, $f_{0i}(N)$, must be expressible in terms of numerically well defined factorized PDF's $f_i(N,\mu_F^2)$, μ_F and the mass singularities to all orders, such that the absence of mass singularities in F_Θ can be made explicit to all orders. Furthermore the dependence of the factorized PDF's on μ_F must be well defined. This is analogous to the renormalization procedure outlined in section 1.1: The $f_{0i}(N)$ are analogous to a_0, the $f_i(N,\mu_F^2)$ are analogous to $a(\mu_R^2)$ and μ_F is analogous to μ_R.

We write equation (2.5.21) in the form

$$F_\Theta(N,Q^2) = \sum_i \widehat{F}_{\Theta i}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) f_{0i}(N). \tag{2.8.1}$$

$\widehat{F}_{\Theta i}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$ are the bare partonic F_Θ cross sections whose truncated perturbative expansions are singular as $\kappa^2 \to 0$. We will see that we can write

$$\widehat{F}_{\Theta i}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) = \sum_j \widetilde{F}_{\Theta j}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right) \Gamma_{ji}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) \tag{2.8.2}$$

where the $\Gamma_{ij}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right)$ are functions whose truncated perturbative expansions are singular as $\kappa^2 \to 0$, and where $\widetilde{F}_{\Theta j}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right)$ are functions whose truncated perturbative expansions are finite as $\kappa^2 \to 0$. Then factorized PDF's are given by

$$f_i(N,\mu_F^2) = \sum_j \Gamma_{ij}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) f_{0j}(N,\kappa^2), \tag{2.8.3}$$

which are numerically well defined and have a non singular dependence on μ_F^2. This is called the *factorization theorem* [16, 17, 18]. We can then write

$$F_\Theta(N, Q^2) = \sum_i \tilde{F}_{\Theta i}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right) f_i(N, \mu_F^2). \tag{2.8.4}$$

This equation has the potential to be phenomenologically useful when $Q \sim \mu_F \gg \Lambda_{QCD}$.

We now outline the derivation of the first part of the factorization theorem, equation (2.8.2) and its successive paragraph, following [16] and [18]. The second part of the factorization theorem, that the $f_i(N, \mu_F^2)$ have a non singular dependence on μ_F^2, will be left to section 2.9. This theorem can be generalized for any physical hadron initiated process involving a hard part, but we shall only consider DIS here. Recall equation (2.5.21) is calculated from the diagram in figure 2.1 whose 2PI expansion is shown in figure 2.7. Recall from section 2.5 that the diagrams C_0^Θ and K_0 are free of UV and mass singularities in a physical light-like gauge, and that the mass singularities arise from the integration over the momentum k of the lines joining the kernels.

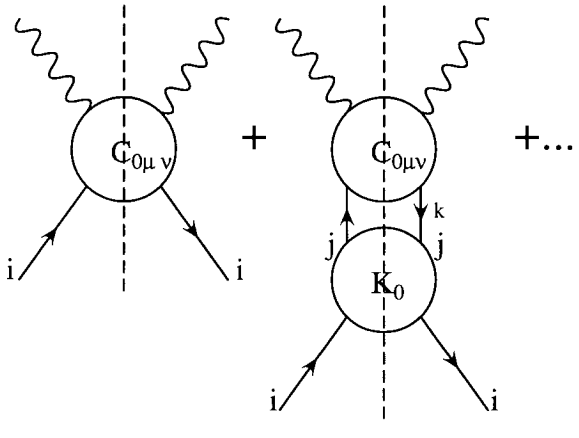

Figure 2.7: Decomposition of the general DIS diagram.

As in section 2.5, parton indices will be omitted from now on. We shall use the Sudakov parameterization for k,

$$k = \beta_k p + k_T + \left(\frac{k^2 - k_T^2}{2\beta_k p \cdot \bar{p}}\right)\bar{p}, \tag{2.8.5}$$

where

$$p^2 = \bar{p}^2 = p \cdot k_T = \bar{p} \cdot k_T = 0 \qquad p \cdot \bar{p} \neq 0. \tag{2.8.6}$$

Thus

$$d^4k = \frac{d\beta_k}{2\beta_k} dk^2 d^2\mathbf{k_T}. \tag{2.8.7}$$

We regularize all dk^2 integrals as before, by replacing their lower bound 0 with a small quantity κ^2. All particles are massless.

As in equation (2.5.5)

$$A \cdot B = \sum_s \int d^4k A^s(v,k) B^s(k,u), \tag{2.8.8}$$

where the suppression of certain spin indices is as in section 2.5. We define a projection operator $\hat{P} = \hat{P} \cdot \hat{P}$ that acts between A and B in the form

$$A \cdot \hat{P} \cdot B = \sum_s \int d^4k \sum_t \int d^4k' A^s(v,k') \hat{P}^{st}(k',k) B^t(k,u). \tag{2.8.9}$$

$\hat{P}$ is diagonal in parton indices, which we have chosen to omit for brevity, is invariant with respect to boosts in the spatial $\mathbf{p}$ direction, and projects out physical states on the lines joining A and B as follows. $\hat{P}$ projects out the physical polarizations, sets $k = \beta_k p$ in A and projects out the mass singularity of the dk^2 integral from κ^2 to ∞ in B. Thus, temporarily restoring parton indices i and j,

$$\hat{P}_{ij}^{rt}(k',k) = \begin{cases} \delta^{(4)}(k' - \beta_k p) w\left(\frac{k^2}{\mu_F^2}, \beta_k\right) & \text{if } r = t = \text{physical and } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8.10}$$

It is necessary to introduce a mass scale μ_F^2 to allow w to depend on k^2 whilst being dimensionless. w must obey

$$w(0,\beta) = 1. \tag{2.8.11}$$

This condition ensures the mass singularities are properly projected out, and hence ensures that $\hat{P}\cdot\hat{P} = \hat{P}$. Now

$$A\cdot\hat{P}\cdot B = \sum_{s=\text{phys}} \int \frac{d\beta_k}{2\beta_k} \int_{\kappa^2}^{\infty} dk^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k_T} w\left(\frac{k^2}{\mu_F^2},\beta_k\right) A^s(v,\beta_k p) B^s(k,u), \tag{2.8.12}$$

so any term of the form $A\cdot\hat{P}\cdot B$ is of the form of a convolution over a single variable. We will choose w such that if $B^s(k,u)$ is free of UV singularities, so is equation (2.8.12) after the $\mathbf{k_T}$ and k^2 integration. This requires w to fall off sufficiently fast as $k^2 \to \infty$.

Terms of the form $A\cdot(\mathbb{1}-\hat{P})\cdot B$ are free of mass singularities if A and B are free of mass singularities. Explicitly,

$$\begin{aligned} A\cdot(\mathbb{1}-\hat{P})\cdot B = &\sum_{s=\text{unphys}} \int \frac{d\beta_k}{2\beta_k} \int_0^{\infty} dk^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k_T} A^s(v,k) B^s(k,u) \\ &+ \sum_{s=\text{phys}} \int \frac{d\beta_k}{2\beta_k} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k_T} \int_{\kappa^2}^{\infty} dk^2 \Bigg[A^s(v,k) B^s(k,u) \\ &\quad - w\left(\frac{k^2}{\mu_F^2},\beta_k\right) A^s(v,\beta_k p) B^s(k,u)\Bigg]. \end{aligned} \tag{2.8.13}$$

The first term in equation (2.8.13) is a sum over unphysical polarizations only, so it is free of mass singularities. Physically this is because virtual particles can only propagate over a finite time and space, so their zero momentum states are suppressed. Each of the two terms in the square brackets are only singular as both $k_T \to 0$ and $k^2 \to 0$, and here the singularities cancel.

We now use the above results to write $\widehat{F}_\Theta$ as a convolution over a single variable of a diagram that is free of mass singularities with a diagram that contains all

the mass singularities of $\widehat{F}_\Theta$, i.e. to write $\widehat{F}_\Theta$ in the form of equation (2.8.2). We first write

$$\frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot \hat{O} = G \cdot \Gamma, \tag{2.8.14}$$

where

$$G = \frac{\mathbb{1}}{\mathbb{1} - (\mathbb{1} - \hat{P}) \cdot K_0} \cdot \hat{O} \tag{2.8.15}$$

$$\Gamma = \frac{\mathbb{1}}{\mathbb{1} - \hat{P} \cdot K_0 \cdot \frac{\mathbb{1}}{\mathbb{1} - (\mathbb{1} - \hat{P}) \cdot K_0}} \cdot \hat{O}. \tag{2.8.16}$$

Here we have used the fact that $\hat{O} \cdot \hat{P} = \hat{P}$ to place the operator $\hat{O}$ on the bottom of G. Then we can rewrite equation (2.5.21) in the form

$$\widehat{F}_\Theta = \widetilde{F}_\Theta \cdot \Gamma, \tag{2.8.17}$$

where

$$\widetilde{F}_\Theta = C_0^\Theta \cdot G. \tag{2.8.18}$$

Since terms of the form $A \cdot (\mathbb{1} - \hat{P}) \cdot B$ are free of mass singularities if A and B are free of mass singularities, $\widetilde{F}_\Theta$ is free of mass singularities. All mass singularities are contained in Γ. In contrast to equation (2.5.21), where the "$\cdot$" involves sums and four-momentum integrations, equation (2.8.17) is in the form of a convolution over a single variable, since terms of the form $A \cdot \hat{P} \cdot B$ are of the form of convolutions over a single variable.

The freedom to choose the factorization scheme is due to the fact that we are free to redefine the functional form of $w\left(\frac{k^2}{\mu_F^2}, \beta_k\right)$.

The above can be repeated using *dimensional regularization* instead of using a cutoff κ^2. In this case we work in $d = 4 - 2\epsilon$ dimensions, with $\epsilon < 0$, and then we

can set $\kappa = 0$ everywhere since mass singularities vanish in a non integer number of dimensions. Dimensional regularization is used to define the $\overline{MS}$ scheme. The practical procedure for calculating renormalized and factorized quantities in the $\overline{MS}$ scheme is outlined in appendix C.

Alternatively we could have given all partons a small mass to regulate the mass singularities, so we see that $\tilde{F}_\Theta$ is well defined in the massless limit.

Note Γ defined in equation (2.8.16) can be written in the form

$$\Gamma = \left(\mathbb{1} + \hat{P} \cdot K\right) \cdot \hat{O}, \tag{2.8.19}$$

where

$$K = \frac{\mathbb{1}}{\mathbb{1} - K_0} - \mathbb{1} \tag{2.8.20}$$

is just the full sum of Feynman graphs which have an off shell parton at the bottom and an off shell one at the top, i.e. it is the cross section for the fully inclusive process with two off shell partons in the initial state. Using a similar derivation to that of section 1.2 for $W^{\mu\nu}$, and restoring spin and parton indices, we find that when the parton at the bottom of the diagram is in a physical state,

$$\left(K \cdot \hat{O}\right)_{ij}^{rs}(k,p) = \int d^4y e^{ik\cdot y} \langle j_s(p)|\overline{\psi}_i^r(y)\psi_i^r(0)|j_s(p)\rangle. \tag{2.8.21}$$

Γ takes the form

$$\Gamma_{ij}^{rs}(\beta_k) = \delta(1-\beta_k)\delta_{ij}\delta_{rs} + \int_{\kappa^2}^{\infty} dk'^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{k}'_{\mathbf{T}} w\left(\frac{k'^2}{\mu_F^2}, \beta_k\right) K_{ij}^{rs}(k',p)|_{\beta'_k=\beta_k}. \tag{2.8.22}$$

2.9 The DGLAP Equation

The $\widetilde{F}_{\Theta i}$ can only be calculated when $Q^2 = O(\mu_F^2)$, since otherwise $\ln^n \frac{Q^2}{\mu_F^2}$ terms spoil the convergence of its series. A way round this problem is, instead of keeping μ_F constant, to set $\mu_F^2 = O(Q^2)$. Taking the case $\mu_F^2 = Q^2$ for simplicity, and defining

$$\widetilde{F}_{\Theta i}(N, a_s(Q^2)) = \widetilde{F}_{\Theta i}(N, 1, a_s(Q^2)), \tag{2.9.1}$$

equation (2.8.4) becomes

$$F_\Theta(N, Q^2) = \sum_i \widetilde{F}_{\Theta i}(N, a_s(Q^2)) f_i(N, Q^2). \tag{2.9.2}$$

Here we see that we would need to know $f_i(N, Q^2)$ for all N *and* Q^2. It would be beneficial if, given that we know the $f_i(N, \mu_F^2)$ at one value of μ_F^2, we could obtain the $f_i(N, \mu_F^2)$ at any other value of μ_F^2 perturbatively. This can be done, as we will now see.

From equation (2.8.16), using the identity $dO^{-1} = -O^{-1} dO O^{-1}$ for a non-commuting object O, and the fact that only $\hat{P}$ depends on μ_F, we find that an infinitesimal change in μ_F gives a change in Γ of

$$d\Gamma = (d\hat{P}) \cdot K_0 \cdot G \cdot \Gamma. \tag{2.9.3}$$

From the results of section 2.8, we find that $(d\hat{P}) \cdot K_0 \cdot G$ is free of mass singularities. Here we assume that

$$\frac{d}{d \ln \mu_F^2} w\left(\frac{k^2}{\mu_F^2}, \beta\right) = -\frac{d}{d \ln k^2} w\left(\frac{k^2}{\mu_F^2}, \beta\right) \to 0 \text{ as } k^2 \to 0 \tag{2.9.4}$$

sufficiently fast, in particular in the $\overline{MS}$ scheme which we are interested in. Furthermore recall that the $\cdot$ in any term of the form $A \cdot \Gamma$ reduces to just an integration over a single variable and a sum over flavours. Thus we can write

$$\frac{d}{d\ln\mu_F^2}\Gamma_{ij}\left(N,\frac{\mu_F^2}{\kappa^2},a_s(\mu_F^2)\right)=\sum_k a_s(\mu_F^2)P_{ik}(N,a_s(\mu_F^2))\Gamma_{kj}\left(N,\frac{\mu_F^2}{\kappa^2},a_s(\mu_F^2)\right), \tag{2.9.5}$$

where $a_s(\mu_F^2)P_{ik}(N,a_s(\mu_F^2))$, the *splitting functions*, are functions whose truncated perturbative expansions are finite as $\kappa^2\to 0$. A solution to equation (2.9.5) with the normalization $\Gamma_{ij}=\delta_{ij}$ for $\mu_F^2=\kappa^2$, which we denote as Γ'_{ij}, is

$$\Gamma'_{ij}\left(N,\frac{\mu_F^2}{\kappa^2},a_s(\mu_F^2)\right)=\left[P\exp\left(\int_{\kappa^2}^{\mu_F^2}\frac{dq^2}{q^2}a_s(q^2)P(N,a_s(q^2))\right)\right]_{ij}, \tag{2.9.6}$$

where "P" denotes path ordering of the $P(N,a_s(q^2))$ with respect to q^2. Equation (2.9.6) illustrates an example of the explicit κ^2 dependence of Γ_{ij}. In general, Γ_{ij} always has the form

$$\Gamma_{ik}\left(N,\frac{\mu_F^2}{\kappa^2},a_s(\mu_F^2)\right)=\sum_j E_{ij}(N,a_s(\mu_F^2),a_s(\mu_{F0}^2))\Gamma_{jk}\left(N,\frac{\mu_{F0}^2}{\kappa^2},a_s(\mu_{F0}^2)\right), \tag{2.9.7}$$

where

$$E_{ij}(N,a_s(\mu_F^2),a_s(\mu_{F0}^2))=\left[P\exp\left(\int_{\mu_{F0}^2}^{\mu_F^2}\frac{dq^2}{q^2}a_s(q^2)P(N,a_s(q^2))\right)\right]_{ij}. \tag{2.9.8}$$

E_{ij} is called the *evolution kernel*. An alternative way of writing E_{ij} is

$$E_{ij}(N,a_s(\mu_F^2),a_s(\mu_{F0}^2))=\left[P\exp\left(\int_{a_s(\mu_{F0}^2)}^{a_s(\mu_F^2)}da_s\frac{a_sP(N,a_s)}{\beta(a_s)}\right)\right]_{ij}. \tag{2.9.9}$$

This form of E_{ij} resums terms of the form $\ln^n\frac{\mu_F^2}{\mu_{F0}^2}$. Differentiating equation (2.8.3) using equation (2.9.5), we find

$$\frac{\partial}{\partial\ln\mu_F^2}f_i(N,\mu_F^2)=\sum_j a_s(\mu_F^2)P_{ij}(N,a_s(\mu_F^2))f_j(N,\mu_F^2). \tag{2.9.10}$$

Equation (2.9.10) is called the *DGLAP equation* [19]. It is to mass singularities what the equation for β, equation (2.4.2), is to UV singularities, and is important for ensuring that the truncated series is close to the all orders result, i.e. for resumming $\ln^n \frac{Q^2}{\mu_F^2}$ terms: In the solution to equation (2.9.10), given by

$$f_i(N, \mu_F^2) = \sum_j E_{ij}(N, a_s(\mu_F^2), a_s(\mu_{F0}^2)) f_j(N, \mu_{F0}^2), \tag{2.9.11}$$

E_{ij} should be calculable to the same accuracy as $a_s P_{ij}$, at least by solving equation (2.9.10) numerically.

Thus to calculate F_Θ at any Q^2, given that we have set $\mu_F \sim Q$ to ensure terms of the form $\ln^n \frac{Q^2}{\mu_F^2}$ are small to help improve the perturbation series in $\widetilde{F}_{\Theta i}$, all we need to know is the PDF's at a given "initial" scale μ_{F0} and use equation (2.9.11) to *evolve* them to μ_F.

The form of E_{ij} in equations (2.9.8) and (2.9.9) are no good for phenomenological applications - we will obtain a more useful form in section (2.16).

The PDF's at scale μ_{F0}^2 contain all processes of energy less than of $O(\mu_{F0})$, so perturbative QCD cannot be used to calculate them. In other words, the T_0 of diagram 2.4 has proton legs, and there are no Feynman rules for these. Thus we leave these non-perturbative QCD components as free parameters of the model. Of course, there are an infinite number of these free parameters. Therefore we must guess a parameterised functional form in ξ for each of the PDF's, where the number of parameters is finite. This parameterization will be discussed in later chapters. These parameters are then determined by fitting them to experimental data on cross sections. Two groups performing this analysis are the *CTEQ collaboration* [20, 21, 22, 23] and *MRST* [24, 25, 26, 27, 28, 29, 30].

2.10 The Operator Product Expansion

An alternative method of proving the factorization theorem is by use of the *operator product expansion* (OPE) [31, 32]. Here the operator $T\{J_\mu^{em}(y), J_\nu^{em}(0)\}$

(multiplied by some tensor $\Theta_{\mu\nu}$, e.g. the projectors of equations (1.2.19), (1.2.20) and (1.2.22)) of equation (1.2.13) is expanded in terms of "local" operators, i.e. operators that depend only on a single space time point. The expansion is in powers of $\frac{1}{Q^2}$, so that each coefficient is the contribution to the cross section from a given twist. The coefficients of these operators at leading twist, after operator renormalization at scale μ_F^2, are the $\widetilde{F}_{\Theta i}$. Note operator renormalization is independent of Lagrangian renormalization. The proton-proton matrix elements of the leading twist operators are the PDF's, and the anomalous dimensions of the leading twist operators are the splitting functions. The Mellin variable N refers to the rank (or spin) of the operator, which is an even integer.

The OPE is the same as the formalism of section 2.5 when one starts from an alternative form of equation (2.5.20),

$$F_\Theta = C_0^\Theta \cdot \frac{\mathbb{1}}{\mathbb{1} - (\mathbb{1} - \hat{O}) \cdot K_0} \cdot \hat{O} \cdot \hat{O} \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot T_0 + r. \tag{2.10.1}$$

Recall that the first term on the right hand side of equation (2.10.1) is free of mass singularities and UV divergences (although individual parts might not be), since r_Θ and F_Θ are free of mass singularities and UV divergences. We write equation (2.10.1) in the form

$$F_\Theta = F_\Theta^B \cdot f_B + r_\Theta, \tag{2.10.2}$$

where

$$F_\Theta^B = C_0^\Theta \cdot \frac{\mathbb{1}}{\mathbb{1} - (\mathbb{1} - \hat{O}) \cdot K_0} \cdot \hat{O} \tag{2.10.3}$$

$$f_B = \hat{O} \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot T_0. \tag{2.10.4}$$

Note that $\hat{O}$ does a similar job to $\hat{P}$ in that it projects out mass singularities, since these two operators are the same as $k^2 \to 0$. (Compare equations (2.5.11) and (2.8.10).) Thus equation (2.10.3) is free of mass singularities. Equation (2.10.4)

is just the full sum of cut Feynman diagrams with two proton legs at the bottom and two parton legs at the top, but where these initial partons are physical and on-shell, and have momenta given by equation (2.1.1). Thus, analogously to equation (1.2.13), in the collinear frame and with the axial gauge $A^+ = 0$, we have

$$f_{BI}(\xi) = \int \frac{dy^-}{2\pi} e^{-i\xi P^+ y^-} \langle \text{proton}(P)|\overline{\psi}_I(0, y^-, \mathbf{0})\gamma^+ \psi_I(0)|\text{proton}(P)\rangle \quad (2.10.5)$$

$$f_{Bg}(\xi) = \int \frac{dy^-}{2\pi\xi P^+} e^{-i\xi P^+ y^-} \langle \text{proton}(P)|F^+_\mu(0, y^-, \mathbf{0})F^{\mu+}(0)|\text{proton}(P)\rangle. \quad (2.10.6)$$

$\hat{O}$ in equation (2.10.4) creates UV singularities in the f_B. In other words the operators in equations (2.10.5) and (2.10.6) require renormalization. In Mellin space, the operators in equations (2.10.5) and (2.10.6) are the same as those in the OPE. Using dimensional regularization to regularize the UV divergences, one can define renormalized operators as

$$f_i(N, \mu_F^2) = \lim_{\epsilon \to 0} \sum_j Z_{ij}\left(N, \epsilon, a_s(\mu_F^2)\right) f_{Bj}(N), \quad (2.10.7)$$

where μ_F is the operator renormalization scale, and

$$\frac{\partial Z_{ij}}{\partial \ln \mu_F^2} = \sum_k a_s(\mu_F^2) P_{ik} Z_{kj}. \quad (2.10.8)$$

Equation (2.10.2) then reads

$$F_\Theta = \tilde{F}_\Theta \cdot f + r_\Theta, \quad (2.10.9)$$

where

$$\tilde{F}_{\Theta i}\left(N, \frac{\mu_F^2}{Q^2}, a_s(\mu_F^2)\right)$$

$$= \lim_{\epsilon \to 0} \sum_j F^B_{\Theta j} \left(N, \frac{\mu_F^2}{Q^2}, \epsilon, a_s(\mu_F^2) \right) \left(Z^{-1} \right)_{ji} \left(N, \epsilon, a_s(\mu_F^2) \right). \qquad (2.10.10)$$

Note f_i, P_{ij} and $\widetilde{F}_{\Theta i}$ used here are the same as those introduced in section 2.8, or more correctly can be made to be the same by a suitable choice of scheme.

2.11 Physical Interpretation

From the path integral formalism of a quantum theory, the proton can be considered as a composition of partons at all energy scales from zero to infinity. We may consider the bare PDF's as the PDF's of all partons at the minimum scale κ, or maximum "size" $\frac{1}{\kappa}$, in the regularized QFT. Then we may consider PDF's factorized to the scale μ_F^2 via equation (2.8.3) as the PDF's of partons in the hadron at scale μ_F^2, or "size" $\frac{1}{\mu_F}$.

By comparing equations (2.8.2) and (2.8.4), the Γ_{ij} can be interpreted as the PDF of parton i at scale μ_F^2 in parton j at scale κ^2. $\widetilde{F}_{2i}$ can be interpreted as the parton-photon cross section involving only partons above the scale μ_F^2, and hence is perturbatively calculable for μ_F^2 in the perturbative region.

Equation (2.9.5) can be rewritten

$$\Gamma_{ij} \left(\xi, \frac{\mu_F^2 + d\mu_F^2}{\kappa^2}, a_s(\mu_F^2) \right) - \Gamma_{ij} \left(\xi, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2) \right)$$

$$= \int_\xi^1 \frac{d\beta}{\beta} \sum_k a_s(\mu_F^2) P_{ik} \left(\frac{\xi}{\beta}, a_s(\mu_F^2) \right) d \ln \mu_F^2 \Gamma_{kj} \left(\beta, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2) \right). \qquad (2.11.1)$$

The left hand side is the number density at momentum fraction ξ of partons i in the bare parton j between the scales μ_F^2 and $\mu_F^2 + d\mu_F^2$. This should be equal to the sum over k and integration over momentum fraction β of the number density of partons k at scale μ_F^2 in the bare parton j, multiplied by the probability that this parton k (at momentum fraction β of parton j) contains a parton i between scales μ_F^2 and $\mu_F^2 + d\mu_F^2$ at momentum fraction ξ of parton k's momentum. Thus inspection of the right hand side of equation (2.11.1) shows that the probability

that parton k at scale μ_F^2 contains a parton i between scales μ_F^2 and $\mu_F^2 + d\mu_F^2$ carrying a fraction between ξ and $\xi + d\xi$ of the momentum of parton k is

$$a_s(\mu_F^2)P_{ik}(\xi, a_s(\mu_F^2))d\xi d\ln\mu_F^2. \tag{2.11.2}$$

Thus $a_s(\mu_F^2)P_{ij}(\xi, a_s(\mu_F^2))$ is the number density of partons i at momentum fraction ξ and scale μ_F^2 in a parton j at scale μ_F^2. We can calculate the PDF's at the scale $\mu_F^2 + d\mu_F^2$ from the PDF's at the scale μ_F^2 via equation (2.9.10) because partons at the scale $\mu_F^2 + d\mu_F^2$ are generated perturbatively by partons at the scale μ_F^2 when $\mu_F^2 \gg \Lambda_{QCD}$.

The mixing of PDF's as μ_F^2 increases is caused by the following. Suppose we initially observe partons of type i at the scale μ_F^2 to obtain $f_i(N, \mu_F^2)$. Now suppose we then observe all partons at the scale $\mu_F^2 + d\mu_F^2$. Then we are "looking deeper inside" the partons of all types, and so the partons i "inside" the partons of all types contribute to $f_i(N, \mu_F^2 + d\mu_F^2)$.

2.12 Symmetries Of Splitting Functions

In a "physical scheme", i.e. one where scheme dependent quantities have the same symmetries as QCD, the Γ_{ij} can be written in the form shown in table 2.1. The *valence* part of Γ_{ij}, $\Gamma_{qq}^{(V)}$ and $\Gamma_{q\bar{q}}^{(V)}$, arise from kernels where there is a single quark line joining each of the two quarks at the bottom of the diagram to each quark at the top, and the δ_{IJ} arises because flavour is conserved along this line. The *sea* parts of Γ_{ij}, $\Gamma_{qq}^{(S)}$ and $\Gamma_{q\bar{q}}^{(S)}$, arise from kernels that are otherwise. The flavour independence of the Γ_{ij} is a result of the flavour symmetry of the partons. Γ_{ij} also obeys quark-antiquark symmetry. From equation (2.9.5), P_{ij} obeys

$$a_s P_{ij} = \sum_k \left(\frac{d}{d\ln\mu_F^2}\Gamma_{ik}\right)\Gamma_{kj}^{-1} \tag{2.12.1}$$

Thus P_{ij} obeys the same symmetries as Γ_{ij}, and its resulting simplification is shown in table 2.2.

	q	$\bar{q}$	g
q	$\delta_{IJ}\Gamma_{qq}^{(V)} + \Gamma_{qq}^{(S)}$	$\delta_{IJ}\Gamma_{q\bar{q}}^{(V)} + \Gamma_{q\bar{q}}^{(S)}$	Γ_{gq}
$\bar{q}$	$\delta_{IJ}\Gamma_{q\bar{q}}^{(V)} + \Gamma_{q\bar{q}}^{(S)}$	$\delta_{IJ}\Gamma_{qq}^{(V)} + \Gamma_{qq}^{(S)}$	Γ_{gq}
g	Γ_{qg}	Γ_{qg}	Γ_{gg}

Table 2.1: The form of Γ_{ij}.

	q	$\bar{q}$	g
q	$\delta_{IJ}P_{qq}^{(V)} + P_{qq}^{(S)}$	$\delta_{IJ}P_{q\bar{q}}^{(V)} + P_{q\bar{q}}^{(S)}$	P_{gq}
$\bar{q}$	$\delta_{IJ}P_{q\bar{q}}^{(V)} + P_{q\bar{q}}^{(S)}$	$\delta_{IJ}P_{qq}^{(V)} + P_{qq}^{(S)}$	P_{gq}
g	P_{qg}	P_{qg}	P_{gg}

Table 2.2: The form of P_{ij}.

Finally note that

$$P_{qq}^{(S)(0)} = P_{q\bar{q}}^{(S)(0)} = 0 \qquad P_{qq}^{(S)(1)} = P_{q\bar{q}}^{(S)(1)} \qquad P_{q\bar{q}}^{(V)(0)} = 0. \tag{2.12.2}$$

This is why we only needed the single quantities $P_{ij}^{(0)} = P_{ij}^{(V)(0)}$ for quarks in our calculation of F_2 in section 2.6.

Thus, in the perturbative region, the n_f massless quarks and the gluon will exhibit $SU(n_f)$ flavour symmetry. Then equation (2.9.10) can be simplified as follows. We define the *singlet*,

$$\Sigma = \sum_{I=1}^{n_f} f_I^+, \tag{2.12.3}$$

the *non singlets*,

$$T_{n^2-1} = \left(\sum_{I=1}^{n} f_I^+\right) - n f_n^+, \qquad n = 2, ..., n_f, \tag{2.12.4}$$

and the *valence quarks*,

$$f_I^- = q_I - \bar{q}_I, \qquad I = 1, ..., n_f, \tag{2.12.5}$$

where

$$f_I^+ = q_I + \bar{q}_I, \qquad I = 1, ..., n_f, \tag{2.12.6}$$

and we shall define

$$f_0^+ = g. \tag{2.12.7}$$

We use the following notation:

$$f_F^+ = \Sigma \qquad\qquad f_G^+ = g. \tag{2.12.8}$$

We define

$$\Gamma_{FF} = \Gamma_{qq}^{(V)} + \Gamma_{q\bar{q}}^{(V)} + n_f(\Gamma_{qq}^{(S)} + \Gamma_{q\bar{q}}^{(S)}) \qquad \Gamma_{FG} = 2n_f\Gamma_{qg} \tag{2.12.9}$$

$$\Gamma_{GF} = \Gamma_{gq} \qquad \Gamma_{GG} = \Gamma_{gg} \tag{2.12.10}$$

$$\Gamma_{(\pm)} = \Gamma_{qq}^{(V)} \pm \Gamma_{q\bar{q}}^{(V)}. \tag{2.12.11}$$

Likewise, we define

$$P_{FF} = P_{qq}^{(V)} + P_{q\bar{q}}^{(V)} + n_f(P_{qq}^{(S)} + P_{q\bar{q}}^{(S)}) \qquad P_{FG} = 2n_f P_{qg} \tag{2.12.12}$$

$$P_{GF} = P_{gq} \qquad P_{GG} = P_{gg} \qquad P_{(\pm)} = P_{qq}^{(V)} \pm P_{q\bar{q}}^{(V)}. \tag{2.12.13}$$

Then we find that

$$\frac{\partial}{\partial \ln \mu_F^2}\Gamma_{\alpha\beta} = \sum_\gamma a_s P_{\alpha\gamma}\Gamma_{\gamma\beta} \tag{2.12.14}$$

$$\frac{\partial}{\partial \ln \mu_F^2}\Gamma_{(\pm)} = a_s P_{(\pm)}\Gamma_{(\pm)}, \tag{2.12.15}$$

where Greek indices take on the values F and G. Now it is easy to show that

$$f_\alpha^+ = \sum_\beta \Gamma_{\alpha\beta} f_{0\beta}^+ \tag{2.12.16}$$

$$T_{n^2-1} = \Gamma_+ T_{0\ n^2-1} \qquad f_I^- = \Gamma_- f_{0I}^-. \tag{2.12.17}$$

Thus the DGLAP equation can be split up into the equations

$$\frac{\partial}{\partial \ln \mu_F^2} f_\alpha^+(N,\mu_F^2) = \sum_{\alpha\beta} a_s(\mu_F^2) P_{\alpha\beta}(N, a_s(\mu_F^2)) f_\beta^+(N,\mu_F^2) \tag{2.12.18}$$

$$\frac{\partial}{\partial \ln \mu_F^2} T_{n^2-1}(N,\mu_F^2) = a_s(\mu_F^2) P_{(+)}(N, a_s(\mu_F^2)) T_{n^2-1}(N,\mu_F^2) \tag{2.12.19}$$

$$\frac{\partial}{\partial \ln \mu_F^2} f_I^-(N,\mu_F^2) = a_s(\mu_F^2) P_{(-)}(N, a_s(\mu_F^2)) f_I^-(N,\mu_F^2). \tag{2.12.20}$$

The singlet splitting functions are singular as $N \to 1$. Physically, this is because the gluons are bosons and therefore the number density of gluons in the ground state is infinite, and then the fact that they can emit quarks and antiquarks gives rise to an infinite number of these. In contrast, the non-singlet splitting functions in fact vanish as $N \to 1$ because the gluons obey quark-antiquark and flavour symmetry, and so the ground state bosons cannot change the valence quark or non singlet content of the proton. The non-singlet splitting functions are singular as $N \to 0$.

2.13 Symmetries Of Partonic Cross Sections

Labelling the electroweak process as V, the $\widehat{F}_{\Theta i}$ can be written in the form

$$\widehat{F}_{\Theta I}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)=K^V_{\Theta I}\hat{C}_{\Theta qNS}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)$$
$$+\langle K^V_\Theta\rangle_{n_f}\left(\hat{C}_{\Theta qS}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)-\hat{C}_{\Theta qNS}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)\right) \qquad (2.13.1)$$

$$\widehat{F}_{\Theta g}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)=\langle K^V_\Theta\rangle_{n_f}\hat{C}_{\Theta g}\left(N,\frac{Q^2}{\kappa^2},a_s(Q^2)\right), \qquad (2.13.2)$$

where the $K^V_{\Theta I}$ are the relevant electroweak factors required for process V, and

$$\langle K^V_\Theta\rangle_{n_f}=\frac{\sum_I K^V_{\Theta I}}{2n_f}. \qquad (2.13.3)$$

(Recall $\sum_I$ refers to a sum over the n_f quarks *and* the n_f antiquarks.) The form of $\widehat{F}_{\Theta I}$ in equation (2.13.1) comes from consideration of the diagrams that make up equation (2.8.18): $K^V_{\Theta I}\hat{C}_{\Theta qNS}$ comes from the (imaginary part of) diagrams of the form in figure 2.8, in which there is a single quark line connecting the incoming quark at the scale μ_F^2 with the quark-quark-photon vertex, and hence there is no sum over quark flavours. The blob in figure 2.8 represents an undrawn part of the diagram, not a Green's function, so e.g. note it does not always contain a vertex or vertices. $\langle K^V_\Theta\rangle_{n_f}\left(\hat{C}_{\Theta qS}-\hat{C}_{\Theta qNS}\right)$ comes from all diagrams that do not fall into this category, and hence there is a sum over quark flavours at the quark-quark-photon vertex since in this case the quark at the quark-quark-photon vertex is always in a loop. A factor n_f is absorbed into $\left(\hat{C}_{\Theta qS}-\hat{C}_{\Theta qNS}\right)$. $\langle K^V_\Theta\rangle_{n_f}\hat{C}_{\Theta g}$ comes from diagrams in which the incoming parton lines are gluons. In these cases the quark at the quark-quark-photon vertex is always in a loop, and hence there is a sum over flavours. Again a factor n_f is absorbed into $\hat{C}_{\Theta g}$.

For F_2 in virtual photon exchange, we have simply

$$K^\gamma_{2I}=K^\gamma_{2\bar{I}}=e^2_{q_I}. \qquad (2.13.4)$$

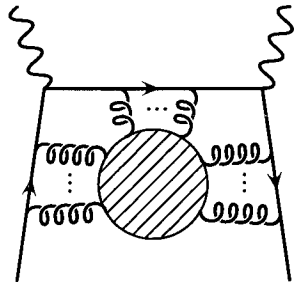

Figure 2.8: Diagrams from which $\hat{C}_{\Theta qNS}$ is obtained.

$K_{2I}^{\gamma,Z}$ is given by the right hand side of equation (2.3.9). For F_3 in neutral current scattering, we have

$$K_{3I}^{\gamma,Z} = -2e_{q_I}A_lA_{q_I}P_Z + 4V_lA_lV_{q_I}A_{q_I}P_Z^2 \tag{2.13.5}$$

$$K_{3\bar{I}}^{\gamma,Z} = -(-2e_{q_I}A_lA_{q_I}P_Z + 4V_lA_lV_{q_I}A_{q_I}P_Z^2), \tag{2.13.6}$$

so that, from equations (2.13.1) and (2.13.2),

$$\widehat{F}_{3I}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$$

$$= (-2e_{q_I}A_lA_{q_I}P_Z + 4V_lA_lV_{q_I}A_{q_I}P_Z^2)\hat{C}_{3qNS}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) \tag{2.13.7}$$

$$\widehat{F}_{3\bar{I}}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$$

$$= -(-2e_{q_I}A_lA_{q_I}P_Z + 4V_lA_lV_{q_I}A_{q_I}P_Z^2)\hat{C}_{3qNS}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) \tag{2.13.8}$$

$$\widehat{F}_{30} = 0. \tag{2.13.9}$$

2.14 Coefficient Functions

For $F_\Theta = F_{1,2}$,

$$\sum_{I=1}^{n_f} \widehat{F}_{\Theta I} = n_f \langle K_\Theta^V \rangle_{n_f} \hat{C}_{\Theta qS}, \tag{2.14.1}$$

so from this result, equations (2.8.2), (2.13.1) and table 2.1, we find in a physical scheme that we can write

$$\hat{C}_{\Theta qS} = C_{\Theta qS}\Gamma_{FF} + C_{\Theta g}\Gamma_{GF}, \tag{2.14.2}$$

where the $C_{\Theta i}$ are free of mass singularities. Likewise we can write

$$\hat{C}_{\Theta g} = C_{\Theta qS}\Gamma_{FG} + C_{\Theta g}\Gamma_{GG}. \tag{2.14.3}$$

Defining

$$\widehat{F}_{\Theta F} = \langle K_\Theta^V \rangle_{n_f} \hat{C}_{\Theta qS} \qquad \widehat{F}_{\Theta G} = \langle K_\Theta^V \rangle_{n_f} \hat{C}_{\Theta g} \tag{2.14.4}$$

$$\widetilde{F}_{\Theta F} = \langle K_\Theta^V \rangle_{n_f} C_{\Theta qS} \qquad \widetilde{F}_{\Theta G} = \langle K_\Theta^V \rangle_{n_f} C_{\Theta g}, \tag{2.14.5}$$

equations (2.14.2) and (2.14.2) can be written

$$\widehat{F}_{\Theta\alpha} = \sum_\beta \widetilde{F}_{\Theta\beta}\Gamma_{\beta\alpha}. \tag{2.14.6}$$

Using the result

$$n_f \widehat{F}_{\Theta I} - \sum_{J=1}^{n_f} \widehat{F}_{\Theta J} = n_f \left(K_{\Theta I}^V - \langle K_\Theta^V \rangle_{n_f} \right) \hat{C}_{\Theta qNS}, \tag{2.14.7}$$

equations (2.8.2), (2.13.1) and (2.14.10) and table 2.1, we can write

$$\hat{C}_{\Theta qNS} = C_{\Theta qNS}\Gamma_+. \tag{2.14.8}$$

For F_3, we can write

$$\hat{C}_{3qNS} = C_{3qNS}\Gamma_-. \tag{2.14.9}$$

The $\tilde{F}_{\Theta i}$ can now be decomposed into the forms, setting $\mu_F = Q$,

$$\begin{aligned}\tilde{F}_{\Theta I}(N, a_s(Q^2)) &= K^V_{\Theta I} C_{\Theta qNS}(N, a_s(Q^2)) \\ +\langle K^V_\Theta \rangle_{n_f} &\left(C_{\Theta qS}(N, a_s(Q^2)) - C_{\Theta qNS}(N, a_s(Q^2))\right)\end{aligned} \tag{2.14.10}$$

$$\tilde{F}_{\Theta g}\left(N, a_s(Q^2)\right) = \langle K^V_\Theta \rangle_{n_f} C_{\Theta g}(N, a_s(Q^2)), \tag{2.14.11}$$

For F_3 in neutral current exchange, equations (2.13.7) to (2.13.9) give

$$F_3(N, Q^2) = \sum_{I=1}^{n_f} K^{\gamma,Z}_{3I} C_{3qNS} f^-_I\,. \tag{2.14.12}$$

$C_{\Theta qS}$ is called the *singlet coefficient function*, $C_{\Theta qNS}$ the *non singlet coefficient function*, $C_{\Theta qS} - C_{\Theta qNS}$ the *pure singlet coefficient function* and $C_{\Theta g}$ the *gluon coefficient function*. Note that

$$C^{(0)}_{\Theta qS} = C^{(0)}_{\Theta qNS} \qquad C^{(1)}_{\Theta qS} = C^{(1)}_{\Theta qNS} \qquad C^{(0)}_{\Theta g} = 0 \tag{2.14.13}$$

for all $\Theta_{\mu\nu}$.

The notation for the $C^{(0,1)}_{2qNS,2g}$ used in our calculation of F_2 in section 2.6, specifically equation (2.7.9), is consistent with this section, and we must take

$$C^{(0)}_{2qNS} = C^{(0)}_{2qS} = 1. \tag{2.14.14}$$

Similarly for F_1 and F_3 we would find

$$C^{(0)}_{1qNS,3qNS} = C^{(0)}_{1qS,3qNS} = 1. \tag{2.14.15}$$

See equations (2.3.15) to (2.3.17) for the charged current scattering case.

From now on, we consider only neutral current exchange, where $K^V_{\Theta I} = K^V_{\Theta \bar{I}}$ for F_1 and F_2, and $K^V_{\Theta I} = -K^V_{\Theta \bar{I}}$ for F_3. From equations (2.3.15) to (2.3.17), our results will also apply to the case that we sum the cross sections for W^+ and W^- exchange.

For $F_\Theta = F_1$ or $F_\Theta = F_2$, we can write F_Θ in terms of singlet and non singlet PDF's using the following method. We write

$$f^+_I = \sum_{J=I}^{n_f} \frac{1}{J}\left(T_{(J+1)^2-1} - T_{J^2-1}\right). \tag{2.14.16}$$

Note $T_{(n_f+1)^2-1} = \Sigma$ since $f^+_{n_f+1} = 0$, and $T_{1^2-1} = 0$ from equation (2.12.4). Using equations (2.9.2), (2.14.10), (2.14.11) and (2.14.16),

$$\begin{aligned} F_\Theta(N,Q^2) = &\left[\sum_{I=2}^{n_f}(\langle K^V_\Theta\rangle_{I-1} - \langle K^V_\Theta\rangle_I) C_{\Theta qNS}(N, a(Q^2)) T_{I^2-1}(N,Q^2)\right] \\ &+\langle K^V_\Theta\rangle_{n_f} C_{\Theta qS}(N, a(Q^2))\Sigma(N,Q^2) \\ &+\langle K^V_\Theta\rangle_{n_f} C_{\Theta g}(N, a(Q^2)) g(N,Q^2). \end{aligned} \tag{2.14.17}$$

2.15 State Of The Art

For a *next to leading order* (NLO) analysis, we would use $P^{(0)}_{ij}$ and $P^{(1)}_{ij}$ in the evolution of the PDF's, and $C^{(0)}_i$ and $C^{(1)}_i$. The reason for this will be seen in section 2.17. The references for the explicit forms of the P_{ij} and C_i are listed in [33], and are typically in the $\overline{MS}$ scheme for both renormalization and factorization. Some of the functions known are

Coefficient functions:

$C^{(1)}_{\Theta qNS}$, $C^{(1)}_{\Theta qS}$ and $C^{(1)}_{\Theta g}$ [34] $C^{(2)}_{\Theta qS}$ and $C^{(2)}_{\Theta qNS}$ [35] $C^{(2)}_{\Theta g}$ [36]

Splitting functions:

$P^{(0)}_{ij}$ [37] $P^{(1)}_{ij}$ [38, 39, 40]

[41] contains the Mellin transforms of typical terms that appear in splitting functions and coefficient functions up to two loops.

The three loop (NNLO) splitting functions are unknown, however some of their even moments have been calculated [42], and an approximate NNLO analysis is underway [43].

2.16 Solving The DGLAP Equation

We now show how to solve equations (2.12.18), (2.12.19) and (2.12.20). We use the following method from [44], where equation (2.9.9) is used. We define

$$R(N, a_s) = \frac{-\beta_0 a_s^3 P(N, a_s)}{\beta(a_s)}, \tag{2.16.1}$$

so that equation (2.9.9) reads

$$E(N, a_s(\mu_F^2), a_s(\mu_{F0}^2)) = P \exp\left(-\frac{1}{\beta_0}\int_{a_s(\mu_{F0}^2)}^{a_s(\mu_F^2)} da_s \frac{1}{a_s} R(N, a_s)\right). \tag{2.16.2}$$

Writing

$$R(N, a_s) = \sum_{i=0}^{\infty} a_s^i R^{(i)}(N), \tag{2.16.3}$$

we have

$$R^{(0)} = P^{(0)} \qquad R^{(n)} = P^{(n)} - \frac{1}{\beta_0}\sum_{i=1}^{n} \beta_i R^{(n-i)}. \tag{2.16.4}$$

For the non-singlets we obtain

$$T_{n^2-1}(N,\mu_F^2) = \left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(P_{(+)}^{(0)}(N)/\beta_0\right)}$$

$$\times \exp\left[-\sum_{i=1}^{\infty}\frac{R_{(+)}^{(i)}(N)}{\beta_0 i}\left(a_s^i(\mu_F^2)-a_s^i(\mu_{F0}^2)\right)\right] T_{n^2-1}(N,\mu_{F0}^2). \tag{2.16.5}$$

$$f_I^-(N,\mu_F^2) = \left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(P_{(-)}^{(0)}(N)/\beta_0\right)}$$

$$\times \exp\left[-\sum_{i=1}^{\infty}\frac{R_{(-)}^{(i)}(N)}{\beta_0 i}\left(a_s^i(\mu_F^2)-a_s^i(\mu_{F0}^2)\right)\right] f_I^-(N,\mu_{F0}^2). \tag{2.16.6}$$

For the singlet, suppose we expand the evolution kernel in the argument of the exponential. We would find to NNLO that

$$\begin{aligned} E = \mathbb{1} &- \frac{R^{(0)}}{\beta_0}\ln\frac{a_s(\mu_F^2)}{a_s(\mu_{F0}^2)} + \frac{1}{2}\left(\frac{R^{(0)}}{\beta_0}\right)^2\ln^2\frac{a_s(\mu_F^2)}{a_s(\mu_{F0}^2)} - \frac{1}{6}\left(\frac{R^{(0)}}{\beta_0}\right)^3\ln^3\frac{a_s(\mu_F^2)}{a_s(\mu_{F0}^2)} \\ &+(a_s(\mu_F^2)-a_s(\mu_{F0}^2))\left(\frac{1}{\beta_0^2}\left(R^{(0)}R^{(1)}-R^{(1)}R^{(0)}\right)-\frac{R^{(1)}}{\beta_0}\right) \\ &+(a_s(\mu_F^2)-a_s(\mu_{F0}^2))^2\left(\frac{R^{(1)2}}{2\beta_0^2}-\frac{R^{(2)}}{2\beta_0}\right) \\ &+\left(R^{(1)}R^{(0)}a_s(\mu_F^2)-R^{(0)}R^{(1)}a_s(\mu_{F0}^2)\right)\ln\frac{a_s(\mu_F^2)}{a_s(\mu_{F0}^2)}. \end{aligned} \tag{2.16.7}$$

However, we wish to have an approximate form for E that gives an accurate picture of the all orders E in the whole range $O(1) > a_s(\mu_F^2), a_s(\mu_{F0}^2) > 0$, since we want to be able to predict cross sections of arbitrarily high energy, and the $\ln^i\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}$ terms above spoil this picture in the limit $a_s \to 0$. Thus these logarithms need to be separated from the rest of the expression and then resummed. This can be done essentially by diagonalization of $R^{(0)}$. We introduce the matrices

$$M^+ = \frac{1}{\lambda_+ - \lambda_-}\left(P^{(0)} - \lambda_-\mathbb{1}\right) \tag{2.16.8}$$

$$M^- = \frac{1}{\lambda_+ - \lambda_-}\left(\lambda_+ \mathbb{1} - P^{(0)}\right), \tag{2.16.9}$$

where

$$\lambda_\pm = \frac{1}{2}\left[P_{FF}^{(0)} + P_{GG}^{(0)} \pm \left(\left(P_{FF}^{(0)} - P_{GG}^{(0)}\right)^2 + 4P_{FG}^{(0)}P_{GF}^{(0)}\right)^{\frac{1}{2}}\right]. \tag{2.16.10}$$

M^+ and M^- are projection matrices, i.e.

$$M^{+2} = M^+ \qquad M^{-2} = M^- \qquad M^+M^- = M^-M^+ = 0. \tag{2.16.11}$$

Furthermore

$$M^+ + M^- = \mathbb{1} \qquad R^{(0)} = \lambda_+ M^+ + \lambda_- M^-. \tag{2.16.12}$$

To lowest order, the solution to equation (2.12.18) can then be written

$$f^+(N,\mu_F^2) = \left(M^+(N)\left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(\frac{\lambda_+(N)}{\beta_0}\right)} + M^-(N)\left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(\frac{\lambda_-(N)}{\beta_0}\right)}\right) f^+(N,\mu_{F0}^2). \tag{2.16.13}$$

This expression is free of logarithms of a_s. It resums all logarithms of this type that are of $O(a_s)$ in E. This motivates us write the general solution to equation (2.12.18) in the form

$$f^+(N,\mu_F^2) = U(N,a_s(\mu_F^2)) \left(M^+(N)\left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(\frac{\lambda_+(N)}{\beta_0}\right)} + M^-(N)\left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\left(\frac{\lambda_-(N)}{\beta_0}\right)}\right) U^{-1}(N,a_s(\mu_{F0}^2)) f^+(N,\mu_{F0}^2), \tag{2.16.14}$$

where $U(N,0) = \mathbb{1}$. For equation (2.16.14) to obey equation (2.12.18), U must obey (omitting the N argument for brevity)

$$U(a_s(\mu_F^2)) = M^+\left(\mathbb{1} - \frac{1}{\beta_0}\int_0^{a_s} daa^{-1}\left[R(a) - R(0)\right]U(a)\right)M^+$$
$$+M^-\left(\mathbb{1} - \frac{1}{\beta_0}a_s^{\left(\frac{\lambda_+-\lambda_-}{\beta_0}\right)}\int_0^{a_s} daa^{-\left(\frac{\lambda_+-\lambda_-}{\beta_0}\right)-1}\left[R(a) - R(0)\right]U(a)\right)M^+$$
$$+M^-\left(\mathbb{1} - \frac{1}{\beta_0}\int_0^{a_s} daa^{-1}\left[R(a) - R(0)\right]U(a)\right)M^-$$
$$+M^+\left(\mathbb{1} - \frac{1}{\beta_0}a_s^{-\left(\frac{\lambda_+-\lambda_-}{\beta_0}\right)}\int_0^{a_s} daa^{\left(\frac{\lambda_+-\lambda_-}{\beta_0}\right)-1}\left[R(a) - R(0)\right]U(a)\right)M^-. \quad (2.16.15)$$

This is sufficient to obtain U order by order: Defining

$$R' = (R - R^{(0)})U, \quad (2.16.16)$$

and writing

$$U(N,a_s) = \sum_{i=0}^{\infty} a_s^i U^{(i)}(N) \qquad U^{(0)}(N) = \mathbb{1}, \quad (2.16.17)$$

so that

$$R'^{(n)} = R^{(n)} + \sum_{i=1}^{n-1} R^{(n-i)}U^{(i)} \qquad R'^{(1)} = R^{(1)}, \quad (2.16.18)$$

we have

$$U^{(n)} = -\frac{1}{\beta_0 n}M^+R'^{(n)}M^+ + \frac{1}{\lambda_+ - \lambda_- - \beta_0 n}M^-R'^{(n)}M^+$$
$$-\frac{1}{\beta_0 n}M^-R'^{(n)}M^- - \frac{1}{\lambda_+ - \lambda_- + \beta_0 n}M^+R'^{(n)}M^-. \quad (2.16.19)$$

To NNLO we find

$$U^{-1}(N, a_s(\mu_{F0}^2)) = \mathbb{1} - a_s(\mu_{F0}^2)U^{(1)} - a_s^2(\mu_{F0}^2)\left(U^{(2)} - U^{(1)2}\right). \tag{2.16.20}$$

Then to NNLO, equation (2.16.14) has the form

$$\begin{aligned} f^+(N,\mu_F^2) = \sum_{k=\pm}\Big(&M^k + a_s(\mu_F^2)U^{(1)}M^k - a_s(\mu_{F0}^2)M^kU^{(1)} \\ &+a_s^2(\mu_F^2)U^{(2)}M^k - a_s(\mu_F^2)a_s(\mu_{F0}^2)U^{(1)}M^kU^{(1)} \\ &+a_s^2(\mu_{F0}^2)\left(M^kU^{(1)2} - M^kU^{(2)}\right)\Big)\left(\frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}\right)^{\frac{\lambda_k}{\beta_0}} f^+(N,\mu_{F0}^2). \end{aligned} \tag{2.16.21}$$

Note that because of the presence of the terms $(a_s(\mu_{F0}^2)/a_s(\mu_F^2))^{\frac{X}{\beta_0}}$ in equations (2.16.5), (2.16.6) and (2.16.21), where $X = P_+^{(0)}, P_-^{(0)}, \lambda_k$ respectively, i.e. because we have resummed the $\ln^i \frac{a_s(\mu_{F0}^2)}{a_s(\mu_F^2)}$ terms, it is not possible to find an analytical and phenomenological solution in x space.

2.17 Changing The Factorization Scheme

$\widehat{F}_{\Theta i}$ is unique, i.e. it does not depend on the factorization scheme, but $\widetilde{F}_{\Theta i}$ and Γ_{ij} are not. In the case $\mu_R = \mu_F = Q$ for simplicity, we may make the following *scheme change*:

$$\Gamma'\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) = \mathbb{S}(N, a_s(Q^2))\Gamma\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) \tag{2.17.1}$$

$$\widetilde{F}'_{\Theta}(N, a_s(Q^2)) = \widetilde{F}_{\Theta}(N, a_s(Q^2))\mathbb{S}^{-1}(N, a_s(Q^2)) \tag{2.17.2}$$

without changing $\widehat{F}_{\Theta}$, according to equation (2.8.4). Indices have again been omitted for brevity, so that all the above quantities are matrices or vectors except in the case of non singlets, where all quantities are just single numbers. If Γ is in a physical scheme, then if we choose $\mathbb{S}$ to have all the QCD symmetries such as flavour and quark-antiquark symmetry, and such that it is non-zero or has a non

zero determinant in the singlet case (but it is otherwise arbitrary), then Γ' will also be in a physical scheme. We now find the relations between the coefficient functions and splitting functions between different schemes. We study terms up to NNLO. Normalizing $\mathbb{S}$ such that $\mathbb{S} = \mathbb{1}$ when $a_s = 0$, we have

$$\mathbb{S}(N, a_s) = \mathbb{1} + \mathbb{S}^{(1)}(N)a_s + \mathbb{S}^{(2)}(N)a_s^2 \tag{2.17.3}$$

$$\mathbb{S}^{-1}(N, a_s) = \mathbb{1} - \mathbb{S}^{(1)}(N)a_s + \left(\mathbb{S}^{(1)2}(N) - \mathbb{S}^{(2)}(N)\right) a_s^2. \tag{2.17.4}$$

The splitting functions are defined by equation (2.9.5). Thus under a scheme change the splitting functions transform, from equations (2.17.1) and (2.9.5), as

$$\begin{aligned} a_s(Q^2)P'(N, a_s(Q^2)) = \frac{d\mathbb{S}(N, a_s(Q^2))}{d\ln Q^2}\mathbb{S}^{-1}(N, a_s(Q^2)) \\ +\mathbb{S}(N, a_s(Q^2))a_s(Q^2)P(N, a_s(Q^2))\mathbb{S}^{-1}(N, a_s(Q^2)). \end{aligned} \tag{2.17.5}$$

Under this transformation for the splitting functions given by equation (2.17.5), we find

$$P'^{(0)}(N) = P^{(0)}(N) \tag{2.17.6}$$

$$P'^{(1)}(N) = P^{(1)}(N) + \mathbb{S}^{(1)}(N)P^{(0)}(N) - P^{(0)}(N)\mathbb{S}^{(1)}(N) - \beta_0\mathbb{S}^{(1)}(N) \tag{2.17.7}$$

$$\begin{aligned} P'^{(2)}(N) = P^{(2)}(N) + \mathbb{S}^{(1)}(N)P^{(1)}(N) - P^{(1)}(N)\mathbb{S}^{(1)}(N) + \mathbb{S}^{(2)}(N)P^{(0)}(N) \\ -P^{(0)}(N)\mathbb{S}^{(2)}(N) + P^{(0)}(N)\mathbb{S}^{(1)2}(N) - \mathbb{S}^{(1)}(N)P^{(0)}(N)\mathbb{S}^{(1)}(N) \\ -2\beta_0\mathbb{S}^{(2)}(N) - \beta_1\mathbb{S}^{(1)}(N) + \beta_0\mathbb{S}^{(1)2}(N), \end{aligned} \tag{2.17.8}$$

and under the transformation in equation (2.17.2) for the $\widetilde{F}_{2i}$ we find

$$\widetilde{F}_{\Theta}^{\prime(0)}(N) = \widetilde{F}_{\Theta}^{(0)}(N) \tag{2.17.9}$$

$$\widetilde{F}_{\Theta}^{\prime(1)}(N) = \widetilde{F}_{\Theta}^{(1)}(N) - \widetilde{F}_{\Theta}^{(0)}(N)\mathbb{S}^{(1)}(N) \tag{2.17.10}$$

$$\widetilde{F}_{\Theta}^{\prime(2)}(N) = \widetilde{F}_{\Theta}^{(2)}(N)$$
$$-\widetilde{F}_{\Theta}^{(1)}(N)\mathbb{S}^{(1)}(N) - \widetilde{F}_{\Theta}^{(0)}(N)\mathbb{S}^{(2)}(N) + \widetilde{F}_{\Theta}^{(0)}(N)\mathbb{S}^{(1)2}(N). \tag{2.17.11}$$

The scheme independence of $P^{(0)}(N)$ and $\widetilde{F}_{\Theta}^{(0)}(N)$ is a consequence of the choice $\mathbb{S}^{(0)}(N) = \mathbb{1}$.

Continuing to all orders with the procedure used to obtain equations (2.17.6) to (2.17.11) shows that if one of the series P or $\widetilde{F}_{\Theta}$ is known to $O(a_s^n)$, it is unnecessary to obtain the other series to a higher order. This is due to the following: Suppose one of the series is known to $O(a_s^{n+1})$, but the other is only known to $O(a_s^n)$, in a given scheme defined to all orders. Suppose we then do a scheme change of the form $\mathbb{S} = \mathbb{1} + a_s^{n+1}\mathbb{S}^{(n+1)}$. Then, as equations (2.17.6) to (2.17.11) indicate, both series gain a dependence on $\mathbb{S}^{(n+1)}$ at $O(a_s^{n+1})$. Note we are free to choose $\mathbb{S}^{(n+1)}$, within reason, where "free" means the theoretical error on F_{Θ} remains unchanged. However, since the $O(a_s^{n+1})$ term of the $O(a_s^n)$ series is unknown, we are free to fix this term at zero while varying $\mathbb{S}^{(n+1)}$, and then we are free to set the $O(a_s^{n+1})$ term of the $O(a_s^{n+1})$ series to zero by a suitable choice of $\mathbb{S}^{(n+1)}$.

2.18 Factorization And Renormalization Scale Dependence

Suppose we have calculated a perturbation series to $O(a_s^n)$, and suppose that this truncated series is a good approximation to the all orders series, i.e. the error is truly of $O(a_s^{n+1})$. We suppose this applies for any n. Now, the all orders series is independent of μ_R. Consequently, if we change μ_R, the change in the truncated

series will be exactly equal and opposite to the change in the rest of the series, and therefore approximately equal and opposite to the change in the $O(a_s^{n+1})$ term. Now the coefficient of a_s^m in the perturbation series is a finite series in $\ln \frac{Q^2}{\mu_R^2}$, and we assume that each coefficient is of a similar order of magnitude to the others. Thus, a change in μ_R subject to the condition

$$\ln \frac{Q^2}{\mu_R^2} = O(1) \tag{2.18.1}$$

will cause a change in the truncated series of $O(a_s^{n+1})$ or less, since this is the change in the $O(a_s^{n+1})$ piece of the all orders series. Therefore varying the scale μ_R between its maximum and minimum allowed by the condition in equation (2.18.1) is a good way of determining if the series does not converge, and imposes a necessary condition on perturbative results. Furthermore, when fitting theoretical parameters to experimental data sets, consistency implies that if working to $O(a_s^n)$ is sufficient for getting a good fit for one value of μ_R allowed by equation (2.18.1), the fit should be equally good for all values of μ_R allowed by equation (2.18.1). Similar reasonings apply to μ_F. Thus we set

$$\mu_F^2 = k_F Q^2 \tag{2.18.2}$$

$$\mu_R^2 = k_R Q^2, \tag{2.18.3}$$

where k_F and k_R are quantities we will vary, subject to the constraint given by equation (2.18.1), as well as a similar one for μ_F, i.e. $\ln k_{F,R} = O(1)$.

The conventional range of examination is $\frac{1}{4} \leq k_{F,R} \leq 4$. It is important to note that varying k_F and k_R in this manner does not necessarily always give the theoretical error in the calculation in question. Rather, if one intends to compare two theories, for example NLO versus NNLO, varying k_F and k_R only gives the relative error. If one theory shows less dependence on k_F and k_R than another, one can infer the former theory has a smaller theoretical uncertainty, whatever that may be. However, note that k_F and k_R set to particular values in one theory

may not mean the same thing as k_F and k_R set to particular values in another theory. It may be the case that, say, $k_R \sim x$ takes a NLO calculation closest to the true result, whereas when the calculation is taken to NNLO, $k_R \sim y$ may be the optimal value, where x and y are very different (e.g. $x < 1$ and $y > 1$ or vice versa). However, it is the overall spread in the calculation when the range $\frac{1}{4} \leq k_{F,R} \leq 4$ is investigated that one is interested in. Finally, it is perhaps reasonable to suggest that the choice $k_R = k_F$ is more physical, i.e. the cross section has a lower theoretical uncertainty, than when $k_R \neq k_F$.

To implement the above, we need to calculate the change in the perturbation series for $\widetilde{F}_{2i}\left(N, \frac{Q^2}{\mu_F^2}, a_s(Q^2)\right)$ when μ_R^2 and μ_F^2 change from the conventional values $\mu_R^2 = \mu_F^2 = Q^2$ to those values given in equations (2.18.2) and (2.18.3), in terms of conventional quantities, where "conventional" refers to the coefficient functions and splitting functions given in the literature, which are presented with $\mu_R^2 = \mu_F^2 = Q^2$. We choose k_F and k_R to be independent of x and Q^2. We omit parton indices for brevity again. From equation (2.8.2), for an arbitrary value of μ_F^2, we have

$$\widetilde{F}_\Theta\left(N, \frac{Q^2}{\mu_F^2}, a_s(Q^2)\right) = \widehat{F}_\Theta\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) \Gamma^{-1}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right). \quad (2.18.4)$$

$\widehat{F}_\Theta\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$ is obtained from the conventional quantities via the equation

$$\widehat{F}_\Theta\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) = \widetilde{F}_\Theta\left(N, a_s(Q^2)\right) \Gamma\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right). \quad (2.18.5)$$

Substituting equation (2.18.5) into equation (2.18.4) gives

$$\widetilde{F}_\Theta\left(N, \frac{Q^2}{\mu_F^2}, a_s(Q^2)\right) = \widetilde{F}_\Theta\left(N, a_s(Q^2)\right)$$
$$\times\Gamma\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) \Gamma^{-1}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right). \quad (2.18.6)$$

Using equation (2.9.7), we find

$$\widetilde{F}_\Theta\left(N,\frac{Q^2}{\mu_F^2},a_s(Q^2)\right)=\widetilde{F}_\Theta\left(N,a_s(Q^2)\right)$$
$$\times E(N,a_s(Q^2),a_s(\mu_{F0}^2))E^{-1}(N,a_s(\mu_F^2),a_s(\mu_{F0}^2)). \qquad (2.18.7)$$

This gives us the μ_F dependence of the $\widetilde{F}_\Theta$ in terms of conventional quantities. We can also obtain the μ_R dependence of $\widetilde{F}_\Theta$ as follows: If we are working to $O(a_s^n)$, we are justified in replacing all occurrences of $a_s(\mu^2)$, where μ^2 is some arbitrary parameter, with

$$a_s(\mu^2)=h_n\left(\frac{\mu^2}{k_RQ^2},a_s(k_RQ^2)\right)+O(a_s^{n+1}), \qquad (2.18.8)$$

where $h_n\left(\frac{\mu^2}{k_RQ^2},a_s(k_RQ^2)\right)$ is the expansion of $a_s(\mu^2)$ in $a_s(k_RQ^2)$ *only* up to $O(a_s^n)$. Then, by substituting equation (2.18.8) into equation (2.18.7), $\widetilde{F}_\Theta$ can be written as a series in $a_s(k_RQ^2)$, in which only terms up to order n in $a_s(k_RQ^2)$ terms are needed:

$$\widetilde{F}_\Theta\left(N,\frac{Q^2}{\mu_F^2},a_s(Q^2)\right)=\widetilde{F}_\Theta\left(N,h_n\left(\frac{1}{k_R},a_s(k_RQ^2)\right)\right)$$
$$\times E\left(N,h_n\left(\frac{k_F}{k_R},a_s(k_RQ^2)\right),a_s(\mu_{F0}^2)\right)E^{-1}\left(N,h_n\left(\frac{1}{k_R},a_s(k_RQ^2)\right),a_s(\mu_{F0}^2)\right)$$
$$+O(a_s^{n+1}). \qquad (2.18.9)$$

Note that each of the four terms in the above equation must be written as truncated series in $a_s(k_RQ^2)$ up to terms of $O(a_s^n)$. Equation (2.18.9) is all we need to determine the k_F and k_R dependence of $\widetilde{F}_\Theta$ in terms of conventional quantities. We now show how this works for $\widetilde{F}_\Theta$, to NNLO. From equation (2.4.10),

$$h_2\left(\frac{1}{k_R},a_s(k_RQ^2)\right)=a_s(k_RQ^2)+a_s^2(k_RQ^2)\beta_0\ln k_R. \qquad (2.18.10)$$

From equation (2.9.8), we have

$$E\left(N,h_n\left(\frac{k_F}{k_R},a_s(k_RQ^2)\right)a_s(\mu_{F0}^2)\right)=\mathbb{1}+P^{(0)}(N)\ln\left(\frac{k_FQ^2}{\mu_{F0}^2}\right)a_s(k_RQ^2)$$

$$+\left[P^{(1)}(N)\ln\left(\frac{k_F Q^2}{\mu_{F0}^2}\right)+\frac{1}{2}P^{(0)}(N)\beta_0\ln\left(\frac{k_F Q^2}{\mu_{F0}^2}\right)\ln\left(\frac{k_R}{k_F}\frac{k_R Q^2}{\mu_{F0}^2}\right)\right.$$
$$\left.+\frac{1}{2}P^{(0)2}(N)\ln^2\left(\frac{k_F Q^2}{\mu_{F0}^2}\right)\right]a_s^2(k_R Q^2)+O(a_s^3). \qquad (2.18.11)$$

$E^{-1}\left(N, h_n\left(\frac{1}{k_R}, a_s(k_R Q^2)\right), a_s(\mu_{F0}^2)\right)$ can be obtained by setting $k_F = 1$ in equation (2.18.11), and then inverting by expanding in $a_s(k_R Q^2)$.

A short calculation gives

$$\tilde{F}_\Theta\left(N, h_n\left(\frac{1}{k_R}, a_s(k_R Q^2)\right)\right) = \tilde{F}_\Theta^{(0)}(N)$$
$$+\tilde{F}_\Theta^{(1)}(N)a_s(k_R Q^2)+\left[\tilde{F}_\Theta^{(1)}(N)\beta_0\ln k_R+\tilde{F}_\Theta^{(2)}(N)\right]a_s^2(k_R Q^2)+O(a_s^3). \qquad (2.18.12)$$

Thus equation (2.18.9) to NNLO leads to

$$\tilde{F}_\Theta\left(N, \frac{Q^2}{\mu_F^2}, a_s(Q^2)\right) = \tilde{F}_\Theta^{(0)}(N)$$
$$+\left[\tilde{F}_\Theta^{(1)}(N)-\tilde{F}_\Theta^{(0)}(N)P^{(0)}(N)\ln k_F\right]a_s(k_R Q^2)$$
$$+\left[\tilde{F}_\Theta^{(2)}(N)+\tilde{F}_\Theta^{(1)}(N)\beta_0\ln k_R-\tilde{F}_\Theta^{(0)}(N)P^{(1)}(N)\ln k_F\right.$$
$$-\tilde{F}_\Theta^{(0)}(N)P^{(0)}(N)\beta_0\ln k_R\ln k_F+\frac{1}{2}\tilde{F}_\Theta^{(0)}(N)P^{(0)}(N)\beta_0\ln^2 k_F$$
$$\left.+\tilde{F}_\Theta^{(0)}(N)P^{(0)2}(N)\ln^2 k_F-\tilde{F}_\Theta^{(1)}(N)P^{(0)}\ln k_F\right]a_s^2(k_R Q^2)+O(a_s^3). \qquad (2.18.13)$$

We now calculate the changes to the evolution matrix E. Now

$$\frac{\partial f(N, k_F Q^2)}{\partial \ln Q^2} = a_s(k_F Q^2)P(N, a_s(k_F Q^2))f(N, k_F Q^2). \qquad (2.18.14)$$

Including a change in the renormalization scale gives

$$\frac{\partial f(N, k_F Q^2)}{\partial \ln Q^2} = h_n\left(\frac{k_F}{k_R}, a_s(k_R Q^2)\right)P\left(N, h_n\left(\frac{k_F}{k_R}, a_s(k_R Q^2)\right)\right)f(N, k_F Q^2). \qquad (2.18.15)$$

Let us define new splitting functions P' such that

$$a_s(k_R Q^2) P'\left(N, \frac{k_F}{k_R}, a_s(k_R Q^2)\right)$$

$$= h_n\left(\frac{k_F}{k_R}, a_s(k_R Q^2)\right) P\left(N, h_n\left(\frac{k_F}{k_R}, a_s(k_R Q^2)\right)\right). \quad (2.18.16)$$

To NLO,

$$a_s(k_R Q^2) P'\left(N, \frac{k_R}{k_F}, a_s(k_R Q^2)\right)$$

$$= P^{(0)}(N) a_s(k_R Q^2) + \left[P^{(1)}(N) + P^{(0)}(N)\beta_0 \ln\frac{k_R}{k_F}\right] a_s^2(k_R Q^2) + O(a_s^3). \quad (2.18.17)$$

Now we know that

$$\frac{\partial f(N, k_F Q^2)}{\partial \ln Q^2} = a_s(k_F Q^2) \sum_{i=0}^{\infty} P^{(i)}(N) a_s^i(k_F Q^2) f(N, k_F Q^2) \quad (2.18.18)$$

leads to the result

$$f(N, k_F Q^2) = E\left(\{P^{(i)}(N)\}, a_s(k_F Q^2), a_s(\mu_{F0}^2)\right) f(N, \mu_{F0}^2), \quad (2.18.19)$$

where $\{P^{(i)}(N)\}$ in E means that these are the splitting functions at each order to be used in the evaluation of E. Thus

$$\frac{\partial \ln E}{\partial \ln Q^2}\left(\left\{P^{(i)}(N)\right\}, a_s(k_F Q^2), a_s(\mu_{F0}^2)\right)$$

$$= a_s(k_F Q^2) \sum_{i=0}^{\infty} a_s^i(k_F Q^2) P^{(i)}(N), \quad (2.18.20)$$

where we define $d\ln E = dE E^{-1}$. We therefore see that

$$\frac{\partial \ln E}{\partial \ln Q^2}\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s(k_R Q^2), a_s(\mu_{F0}^2)\right)$$

$$= a_s(k_R Q^2) \sum_{i=0}^{\infty} a_s^i(k_R Q^2) P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)$$

$$= a_s(k_F Q^2) \sum_{i=0}^{\infty} a_s^i(k_F Q^2) P^{(i)}(N), \tag{2.18.21}$$

where the last step follows from equation (2.18.16). Thus

$$\frac{\partial \ln E}{\partial \ln Q^2}\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s(k_R Q^2), a_s(\mu_{F0}^2)\right)$$

$$= \frac{\partial \ln E}{\partial \ln Q^2}\left(\left\{P^{(i)}(N)\right\}, a_s(k_F Q^2), a_s(\mu_{F0}^2)\right). \tag{2.18.22}$$

Integrating this gives

$$E\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s(k_R Q^2), a_s(\mu_{F0}^2)\right)$$

$$= E\left(\left\{P^{(i)}(N)\right\}, a_s(k_F Q^2), a_s(\mu_{F0}^2)\right) K(N, k_F, k_R, a_s(\mu_{F0}^2)). \tag{2.18.23}$$

To determine K, we use the fact that $E\left(\left\{P^{(i)}(N)\right\}, a_s(k_F Q^2), a_s(\mu_{F0}^2)\right) = 1$ when $k_F Q^2 = \mu_{F0}^2$. Therefore

$$K = E\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right), a_s(\mu_{F0}^2)\right). \tag{2.18.24}$$

Thus

$$f(N, k_F Q^2) = E\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s(k_R Q^2), a_s(\mu_{F0}^2)\right)$$

$$\times E^{-1}\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right), a_s(\mu_{F0}^2)\right) f(N, \mu_{F0}^2). \tag{2.18.25}$$

Finally, to bring k_R and k_F dependence into the starting scale of E, μ_{F0}^2, we take

$$a_s(\mu_{F0}^2) = h_n\left(\frac{k_F}{k_R}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right), \tag{2.18.26}$$

so that

$$f(N,k_F Q^2) = E\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\}, a_s(k_R Q^2), h_n\left(\frac{k_F}{k_R}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right)$$
$$\times E^{-1}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right), h_n\left(\frac{k_F}{k_R}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right) f(N,\mu_{F0}^2). \tag{2.18.27}$$

We will take $\mu_{F0}^2 = k_F Q_0^2$, and define Q_0^2 to be independent of k_F.

2.19 Thresholds

When $m_i^2 \ll Q^2 \ll m_{i+1}^2$, we may assume that the physical world is well approximated by a theory with $n_f = i$ massless quarks. In section 2.5, we saw that neglecting a quark mass m gives an error of $O\left(\frac{m^2}{Q^2}\right)$ when $Q \gg O(m)$, and neglecting a quark of mass m altogether gives an error of $O\left(\frac{Q^2}{m^2}\right)$ when $Q \ll O(m)$. For now we will also assume we can neglect a quark mass m when $Q > O(m)$, and neglect a quark I altogether if $Q < O(m)$. When $Q = O(m)$, this is clearly a bad approximation - this problem will be dealt with in chapter 5.

We therefore need to understand what to do when Q goes from $Q < O(m)$ to $Q > O(m)$, since the number of flavours n_f must increase by one. The results in this section will be given more justification in chapter 5. For now, we simply use physically intuitive arguments to obtain the procedure to use when crossing a threshold, so note that the results in this section do not follow formally from the massless QCD we have considered so far.

For renormalization with n_f flavours, a dimensionless cross section $\sigma(Q^2)$ has the form

$$\sigma(Q^2) = \sum_{i=0}^{\infty} a_s^{\{n_f\}i}(\mu_R^2)\sigma^{\{n_f\}(i)}\left(Q^2, \frac{Q^2}{\mu_R^2}\right). \tag{2.19.1}$$

We are considering a case where there are no other kinematic variables, for brevity. Here the superscript $\{n_f\}$ refers to the number of flavours used. $a_s^{\{n_f\}}(\mu_R^2)$ evolves

according to the β equation, equation (2.4.2), for n_f massless flavours. n_f massless flavours are used when $O(m_{n_f}^2) < Q^2 < O(m_{n_f+1}^2)$. Thus we define a_s by

$$a_s(\mu_R^2) = a_s^{\{n_f\}}(\mu_R^2) \quad \text{if} \quad m'_{n_f} \leq \mu_R < m'_{n_f+1}, \tag{2.19.2}$$

where m'_i is the *threshold* for parton i. m'_i must obey $m'_i = O(m_i)$, but is otherwise arbitrary. If the $\sigma^{\{n_f\}(i)}\left(Q^2, \frac{Q^2}{\mu_R^2}\right)$ are known for n_f flavours, it is trivial to obtain them for n_f+1 flavours. However, the relation between $a_s^{\{n_f\}}(\mu_R^2)$ and $a_s^{\{n_f+1\}}(\mu_R^2)$ must be known before a change of flavours can be done. We assume that

$$a_s^{\{n_f+1\}}(m'^2_{n_f+1}) = a_s^{\{n_f\}}(m'^2_{n_f+1}) \tag{2.19.3}$$

Then $a_s^{\{n_f+1\}}(\mu_R^2)$ can be evolved from $\mu_R^2 = m'^2_{n_f+1}$ using the β equation, equation (2.4.2), for $n_f + 1$ massless flavours.

Likewise we can define thresholds for factorization as $m''_i = O(m_i)$. Then E is calculated by taking, for $m''^2_{n_f} < \mu_F^2 < m''^2_{n_f+1}$ and $m''^2_{n_0} > \mu_{F0}^2 > m''^2_{n_0-1}$,

$$E(N, a_s(\mu_F^2), a_s(\mu_{F0}^2)) = E^{[n_f]}(N, a_s(\mu_F^2), a_s(m''^2_{n_f}))$$

$$\left(\prod_{i=n_f}^{n_0+1} E^{[i-1]}(N, a_s(m''^2_i), a_s(m''^2_{i-1}))\right) E^{[n_0-1]}(N, a_s(m''^2_{n_0}), a_s(\mu_{F0}^2)). \tag{2.19.4}$$

The superscript $[i]$ refers to the number of flavours to use, in both factorization and renormalization. We set

$$f_{n_f}^+(N, \mu_F^2) = 0 \quad \text{for} \quad \mu_F^2 \leq m''^2_{n_f}. \tag{2.19.5}$$

From threshold, f_i^+ evolves from zero according to the DGLAP equation for n_f massless flavours. Note $E_{ij}^{[n_f]} = 0$ for $i > n_f$ or $j > n_f$. This is called the zero mass variable flavour number scheme (ZM-VFNS). The *heavy quarks*, c, b and t, are therefore perturbatively generated from the *light partons*, u, d and s, and the gluon, since $m_I \gg \Lambda_{QCD}$ for heavy quarks, and the heavy quarks are only

ever "switched on" when $Q > O(m_I)$. The light partons are not determinable perturbatively, and must be taken as inputs since they are always switched on.

When the starting scale, μ_{F0}, is less than m_c'' (but much greater than m_s'' otherwise perturbation theory does not work), we set the starting PDF's for all heavy quarks to zero, whereas the starting PDF's for the light quarks and gluon are inputs which are non zero. Equivalently, the input distributions can be taken as $g(N, \mu_{F0}^2)$, $\Sigma(N, \mu_{F0}^2)$, $T_{2^2-1}(N, \mu_{F0}^2)$ and $T_{3^2-1}(N, \mu_{F0}^2)$. When we reach the threshold m_c'' on evolving upwards, which we do with 3 massless flavours, we have

$$T_{4^2-1}(N, m_c''^2) = \Sigma(N, m_c''^2). \tag{2.19.6}$$

This is true because the charm PDF $f_c^+(N, m_c''^2) = 0$. Then T_{4^2-1} can be evolved from here in the usual way to higher values of μ_F.

When μ_{F0} is greater than the charm threshold m_c'', the initial PDF for the charm quark is not zero, and must be calculated. This is done as follows: The singlet PDF is evolved down from μ_{F0} to m_c'', with 4 massless flavours, i.e. we use

$$E^{[4]}(N, a_s(m_c''^2), a_s(\mu_{F0}^2)). \tag{2.19.7}$$

Then equation (2.19.6) is used, and T_{4^2-1} is then evolved from m_c'' to μ_{F0} with 4 massless flavours, i.e. using equation (2.19.7) with $m_c''^2$ and μ_{F0}^2 interchanged, and then T_{4^2-1} is included as one of the starting PDF's.

In section 2.18, μ_F^2 and μ_R^2 are functions of Q^2 via equations (2.18.2) and (2.18.3), and therefore of one another. We shall choose the number of flavours in both renormalization and factorization to be determined by $\mu_R^2 = k_R Q^2$, i.e. we shall change the number of flavours everywhere when μ_R^2 crosses a threshold $m_i'^2$, since keeping the number of flavours the same in both renormalization and factorization would seem to be closest to the physical truth. We note therefore that the factorization thresholds become

$$m_i''^2 = \frac{k_F m_i'^2}{k_R}. \tag{2.19.8}$$

Note that this is only a necessary condition if the number of flavours in both renormalization and factorization are to change simultaneously. Then the evolution of $f(N,\mu_{F0}^2)$ into $f(N,k_FQ^2)$ is done as follows: Suppose $m_b'^2 < k_RQ^2 < m_t'^2$, and $\mu_{F0}^2 < \frac{k_F}{k_R}m_c'^2$. Then we first take, using equation (2.18.27),

$$f\left(N,\frac{k_Fm_c'^2}{k_R}\right) = E^{[3]}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(m_c'^2),h_n\left(\frac{k_F}{k_R},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right)$$
$$\times E^{[3]-1}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right),h_n\left(\frac{k_F}{k_R},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right)f(N,\mu_{F0}^2). \tag{2.19.9}$$

We note again that the superscript "$[i]$" refers to the number of flavours to use in both factorization and renormalization. Then we take

$$f\left(N,\frac{k_Fm_b'^2}{k_R}\right) = E^{[4]}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(m_b'^2),h_n\left(\frac{k_F}{k_R},a_s(m_c'^2)\right)\right)$$
$$\times E^{[4]-1}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(m_c'^2),h_n\left(\frac{k_F}{k_R},a_s(m_c'^2)\right)\right)f\left(N,\frac{k_Fm_c'^2}{k_R}\right). \tag{2.19.10}$$

Then, finally,

$$f(N,k_FQ^2) = E^{[5]}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(k_RQ^2),h_n\left(\frac{k_F}{k_R},a_s(m_b'^2)\right)\right)$$
$$\times E^{[5]-1}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(m_b'^2),h_n\left(\frac{k_F}{k_R},a_s(m_b'^2)\right)\right)f\left(N,\frac{k_Fm_b'^2}{k_R}\right). \tag{2.19.11}$$

For the case that $\frac{k_F}{k_R}m_b'^2 > \mu_{F0}^2 > \frac{k_F}{k_R}m_c'^2$, we do the following: We evolve the singlet and gluon down from μ_{F0}^2 to $\frac{k_F}{k_R}m_c'^2$ using

$$\begin{pmatrix}\Sigma\\ G\end{pmatrix}\left(N,\frac{k_Fm_c'^2}{k_R}\right) = E^{[4]}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s(m_c'^2),h_n\left(\frac{k_F}{k_R},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right)$$
$$\times E^{[4]-1}\left(\left\{P^{(i)\prime}\left(N,\frac{k_F}{k_R}\right)\right\},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right),h_n\left(\frac{k_F}{k_R},a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right)\right)\right)\begin{pmatrix}\Sigma\\ G\end{pmatrix}(N,\mu_{F0}^2). \tag{2.19.12}$$

Then one takes

$$T_{4^2-1}\left(N, \frac{k_F m_c'^2}{k_R}\right) = \Sigma\left(N, \frac{k_F m_c'^2}{k_R}\right), \tag{2.19.13}$$

since $f_c\left(N, \frac{k_F m_c'^2}{k_R}\right) = 0$, and then T_{4^2-1} is evolved up according to

$$T_{4^2-1}(N, \mu_{F0}^2) = E^{[4]}\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s\left(\frac{k_R}{k_F}\mu_{F0}^2\right), h_n\left(\frac{k_F}{k_R}, a_s(m_c'^2)\right)\right)$$
$$\times E^{[4]-1}\left(\left\{P^{(i)\prime}\left(N, \frac{k_F}{k_R}\right)\right\}, a_s(m_c'^2), h_n\left(\frac{k_F}{k_R}, a_s(m_c'^2)\right)\right) T_{4^2-1}\left(N, \frac{k_F m_c'^2}{k_R}\right). \tag{2.19.14}$$

Then $T_{4^2-1}(N, \mu_{F0}^2)$ is taken as one of the initial PDF's.

2.20 Hadron Mass

We now investigate the effects of the mass of the initial hadron in fully inclusive DIS processes, using [45]. We write 4-vectors as $V^\mu = (V^+, V^-, \mathbf{V})$, where $V^\pm = \frac{1}{\sqrt{2}}(V^0 \pm V^3)$ and $\mathbf{V} = (V^1, V^2)$. In the collinear frame, we have

$$P = \left(P^+, \frac{M^2}{2P^+}, \mathbf{0}\right) \tag{2.20.1}$$

$$q = \left(-\eta P^+, \frac{Q^2}{2\eta P^+}, \mathbf{0}\right), \tag{2.20.2}$$

where M is the mass of the hadron target, and P^+ is arbitrary. η is the *Nachtmann scaling variable.* It is fixed by the relation $x = \frac{Q^2}{2P\cdot q}$, so we get

$$\eta = \frac{x}{\frac{1}{2} + \frac{1}{2}\rho}, \tag{2.20.3}$$

where

$$\rho^2 = 1 + \frac{4M^2x^2}{Q^2}. \tag{2.20.4}$$

Note also that now

$$x = \frac{1}{1 + \frac{(p_X^2 - M^2)}{Q^2}}, \tag{2.20.5}$$

where p_X is the momentum of the final hadronic state. Since $0 \leq p_X^2 \leq \infty$, we find

$$0 \leq x \leq \frac{1}{1 - \frac{M^2}{Q^2}}. \tag{2.20.6}$$

Thus from equation (2.20.3),

$$0 \leq \eta \leq 1. \tag{2.20.7}$$

In the collinear frame, since $\mathbf{p_T}$ can be set to zero for leading twist calculations, we can choose $\hat{O}$ of section 2.5 to set incoming parton momenta to

$$p = (\xi P^+, 0, \mathbf{0}). \tag{2.20.8}$$

Note that $\hat{O}$ is no different from that in the case that the hadron mass is neglected: $\hat{O}$ is again given by equation (2.5.11), where P in this equation, as well as $\bar{P}$, is as in equation (2.5.9) and the parameterization of p is as in equation (2.5.7).

Since the results of section 2.5 apply with or without hadron mass, equation (2.1.3) is still true except for the limits of the ξ integration. Since the + component of the final (real) state created by the initial virtual photon-parton state must be positive, equations (2.20.2) and (2.1.1) give $\xi \geq \eta$. The upper limit of the ξ integration is, as before, 1. This follows from the fact that the + component of the momentum of what is left of the proton after the parton has left it, $(1-\xi)P^+$, must be positive. Thus we have

$$W_{\mu\nu}(P,q) = \sum_i \int_\eta^1 \frac{d\xi}{\xi} W^i_{\mu\nu}(p,q) f_{0i}(\xi). \tag{2.20.9}$$

From equations (2.20.9), (1.2.18) and (1.2.21), we find the analogues of equations (1.2.19), (1.2.20) and (1.2.22) to be

$$F_1 = \left(\frac{2x^2}{Q^2\rho^2} P^\mu P^\nu - \frac{1}{2} g^{\mu\nu} \right) W_{\mu\nu} \tag{2.20.10}$$

$$F_2 = \frac{1}{\rho^4} \left(\frac{12x^3}{Q^2} P^\mu P^\nu - x\rho^2 g^{\mu\nu} \right) W_{\mu\nu} \tag{2.20.11}$$

$$F_3 = \frac{2x}{\rho^2 Q^2} i\epsilon^{\mu\nu\alpha\beta} P_\alpha q_\beta W_{\mu\nu} \tag{2.20.12}$$

respectively. We now use these projection operators and equation (2.20.9) to find the dependence of F_1, F_2 and F_3 on the $\widehat{F}_{1i}$, $\widehat{F}_{2i}$ and $\widehat{F}_{3i}$. The conventional $\widehat{F}_{\Theta i}$ as calculated in the literature without hadron mass must be modified to include hadron mass. Treating $\widehat{F}_{\Theta i}$ as a function of p and q, this functional dependence should not change when we include the hadron mass, since $W^i_{\mu\nu}$ has nothing to do with hadron mass. Physically, the parton that has left the proton does not "see" the hadron mass. Since the $\widehat{F}_{\Theta i}$ given in the literature are calculated with p given by equation (2.20.8) and q given by equation (2.20.2) but with η replaced by x, to modify $\widehat{F}_{2i}$ in the presence of hadron mass we simply change x to η in $\widehat{F}_{2i}$. Thus, acting on $W^i_{\mu\nu}$, as given by equation (2.1.9), with the tensors in equations (2.20.10) to (2.20.12), and then using equation (2.20.9), we find

$$F_1(\eta, Q^2) = \sum_{i=0}^{n_f} \int_\eta^1 \frac{d\xi}{\xi} \widehat{F}_{1i}\left(\frac{\eta}{\xi}, Q^2\right) f^+_{0i}(\xi) \tag{2.20.13}$$

$$F_2(\eta, Q^2) = \Omega(\eta, Q^2) \sum_{i=0}^{n_f} \int_\eta^1 d\xi \widehat{F}_{2i}\left(\frac{\eta}{\xi}, Q^2\right) f^+_{0i}(\xi) \tag{2.20.14}$$

$$F_3(\eta, Q^2) = \frac{1}{\rho} \sum_{I=1}^{n_f} \int_\eta^1 \frac{d\xi}{\xi} \widehat{F}_{3I}\left(\frac{\eta}{\xi}, Q^2\right) f^-_{0I}(\xi), \tag{2.20.15}$$

where

$$\Omega(\eta, Q^2) = \frac{\left(1 - \frac{M^2\eta^2}{Q^2}\right)}{\left(1 + \frac{M^2\eta^2}{Q^2}\right)^2}. \tag{2.20.16}$$

Notice it is more convenient to use the kinematic variables (η, Q^2) rather than (x, Q^2). Two useful equations are

$$x = \frac{\eta}{1 - \frac{M^2\eta^2}{Q^2}} \tag{2.20.17}$$

$$\rho = \frac{1 + \frac{M^2\eta^2}{Q^2}}{1 - \frac{M^2\eta^2}{Q^2}}. \tag{2.20.18}$$

2.21 Sum Rules

Equation (2.4.1) was inferred from the physical picture of the parton model. Including the gluon, in N space we expect that

$$g_0(2) + \Sigma_0(2) = 1. \tag{2.21.1}$$

After factorization, the result

$$g(2, \mu_F^2) + \Sigma(2, \mu_F^2) = 1 \tag{2.21.2}$$

follows from equation (2.21.1) if and only if, using equation (2.12.16),

$$\Gamma_{FF}\left(2, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) + \Gamma_{GF}\left(2, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) = 1 \tag{2.21.3}$$

$$\Gamma_{FG}\left(2, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) + \Gamma_{GG}\left(2, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) = 1. \tag{2.21.4}$$

Equation (2.21.2) is called the *momentum sum rule*. For a scheme where the factorized PDF's have a physical interpretation (see section 2.11), equations (2.21.3) and (2.21.4) are what we would expect. Equation (2.21.2) holds for all values of μ_F^2 in a physical scheme. This is called *momentum conservation*, and holds if and only if equations (2.21.3) and (2.21.4) hold for all values of μ_F^2. This is true if and only if, from equation (2.12.18),

$$P_{FF}^{(n)}(2) + P_{GF}^{(n)}(2) = 0 \tag{2.21.5}$$

$$P_{FG}^{(n)}(2) + P_{GG}^{(n)}(2) = 0 \tag{2.21.6}$$

for all n. Momentum conservation holds in the $\overline{MS}$ scheme.

The above can be derived formally from QCD using equations (2.10.5) and (2.10.6). Then in N space

$$g_B(2) + \Sigma_B(2) = 1, \tag{2.21.7}$$

since equation (2.21.7) is the normalized proton-proton matrix element of the momentum operator. This operator is independent of the operator renormalization scheme, since this operator corresponds to an observable, so equation (2.21.2) follows from equation (2.21.7). The $P_{\alpha\beta}(2, a_s(\mu_F^2)$ are the anomalous dimensions of the gluon and flavour-summed quark momentum operator, and these two operators mix under renormalization. The physical reasons for this mixing are given in the last paragraph of section 2.11. However, they are such that the total renormalized momentum operator does not change as μ_F changes, i.e. they obey equations (2.21.5) and (2.21.6).

If the momentum sum rule holds in one scheme, it will hold in another scheme if and only if

$$\mathbb{S}_{FF}(2, a_s) + \mathbb{S}_{GF}(2, a_s) = 1 \tag{2.21.8}$$

$$\mathbb{S}_{FG}(2, a_s) + \mathbb{S}_{GG}(2, a_s) = 1. \tag{2.21.9}$$

In a physical scheme, the *valence sum rule* follows from the fact that the quark quantum numbers of a hadron are always integers:

$$q_I^-(1, \mu_F^2) = n_I. \tag{2.21.10}$$

For example, the proton has $n_1 = 1$, $n_2 = 2$ and $n_I = 0$ for $I > 2$. Conservation of the valence sum rule requires

$$P_{(-)}^{(n)}(1) = 0. \tag{2.21.11}$$

This is called *flavour conservation*, and holds in the $\overline{MS}$ scheme. Note $P_{(-)}(1)$ is the anomalous dimension of each of the quark number operators. Thus in a cross section that depends only on valence quark PDF's, since the cross section is observable and hence renormalization group independent, so therefore is the $N = 1$ factorized partonic cross section, and so this latter quantity must be the same in all physical schemes. In a general physical scheme,

$$\begin{aligned} F_2^{\nu}(0, Q^2) - F_2^{\bar{\nu}}(0, Q^2) &= \sum_{I' \leq n_f} \sum_{J'' \leq n_f} |V_{J''I'}|^2 C_{2qNS}(1, Q^2)(q_{I'}^-(1, Q^2) - q_{J''}^-(1, Q^2)) \\ &= C_{2qNS}(1, Q^2) \sum_{I' \leq n_f} \sum_{J'' \leq n_f} |V_{J''I'}|^2 (n_{I'} - n_{J''}). \end{aligned} \tag{2.21.12}$$

Thus $C_{2qNS}(1, Q^2)$ is the same for all physical schemes. In the DIS scheme, which is a physical scheme,

$$C_{2qNS}(1, Q^2) = 1, \tag{2.21.13}$$

and so equation (2.21.13) is true in all other physical schemes. Equation (2.21.13) is called the *Adler sum rule.* Alternatively we can write

$$C^{(0)}_{2qNS}(1) = 1 \qquad C^{(i)}_{2qNS}(1) = 0 \text{ for } i \geq 1. \qquad (2.21.14)$$

Chapter 3

Drell-Yan

This chapter gives a straightforward extension of the results of the previous chapters to the Drell-Yan process.

3.1 Introduction

The Drell-Yan process is given by

$$h_1 + h_2 \to \text{virtual vector boson} + X, \tag{3.1.1}$$

where the h_α are hadrons, and again X is the unmeasured hadronic final state. The virtual vector boson decays into an observed real lepton-antilepton final state. The diagram for this process is shown in figure 3.1.

The kinematic variables we will use are

$$Q^2 = q^2 \tag{3.1.2}$$

$$\tau = \frac{q^2}{(P_1 + P_2)^2}. \tag{3.1.3}$$

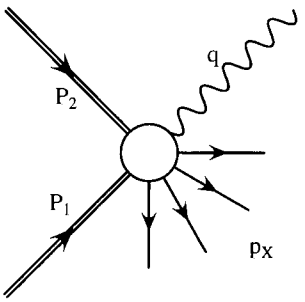

Figure 3.1: General Drell-Yan process.

The limits on these quantities are

$$Q^2 \geq 0 \tag{3.1.4}$$

$$0 \leq \tau \leq 1, \tag{3.1.5}$$

which is due to the following: Since the vector boson decays into two real leptons, which we will take to be massless, of momenta k_1 and k_2, we have

$$q^2 = (k_1 + k_2)^2 = 2k_1 \cdot k_2 = 2E_1E_2(1 - \cos\theta), \tag{3.1.6}$$

where E_1 and E_2 are the energies of the leptons, and θ is the angle between them. Thus equation (3.1.4) follows.

Now we derive equation (3.1.5). From momentum conservation,

$$P_1 + P_2 = p_X + q. \tag{3.1.7}$$

Thus equation (3.1.3) becomes

$$\tau = \frac{q^2}{q^2 + p_X^2 + 2p_X \cdot q}. \tag{3.1.8}$$

Now $q^2, p_X^2 \geq 0$, so it follows that $2p_X \cdot q \geq 0$. Thus, from equation (3.1.8), $\tau \leq 1$ with the equality when $p_X = 0$, and $\tau \geq 0$ with equality when $p_X^2 = \infty$.

We consider the quantity

$$Q^2 \frac{d\sigma_{V \to l_1 \bar{l}_2}}{dQ^2} = \tau \sigma_{0V}(Q^2) W_V(\tau, Q^2), \tag{3.1.9}$$

where the σ_{0V} are the point-like cross sections given by [46]:

$$\sigma_{0Z} = \frac{\pi \alpha_{em}}{4 M_Z \sin^2\theta_W \cos^2\theta_W} \frac{Q^2}{N_c} \frac{\Gamma_{Z \to l\bar{l}}}{(Q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \tag{3.1.10}$$

$$\sigma_{0W} = \frac{\pi \alpha_{em}}{M_W \sin^2\theta_W} \frac{Q^2}{N_c} \frac{\Gamma_{W \to l\bar{\nu}_l}}{(Q^2 - M_W^2)^2 + M_W^2 \Gamma_W^2}. \tag{3.1.11}$$

$\Gamma_{V \to l_1 \bar{l}_2}$ is the decay width of V to leptons, and Γ_V is the total decay width (i.e. to leptons and hadrons). Since $\sin^2\theta_W$ has large radiative corrections, it is better to substitute

$$\sin^2\theta_W = \frac{\pi \alpha_{em}}{G_F \sqrt{2} M_W^2} \tag{3.1.12}$$

so that, using $M_W^2 = M_Z^2 \cos^2\theta_W$,

$$Q^2 \frac{d\sigma_{Z \to l\bar{l}}}{dQ^2} = \frac{\sqrt{2} G_F M_Z}{4} \frac{Q^2}{N_c} \frac{\Gamma_{Z \to l\bar{l}}}{(Q^2 - M_Z^2)^2 + M_Z^2 \Gamma_Z^2} \tau W_Z(\tau, Q^2) \tag{3.1.13}$$

$$Q^2 \frac{d\sigma_{W \to l\bar{\nu}_l}}{dQ^2} = \sqrt{2} G_F M_W \frac{Q^2}{N_c} \frac{\Gamma_{W \to l\bar{\nu}_l}}{(Q^2 - M_W^2)^2 + M_W^2 \Gamma_W^2} \tau W_W(\tau, Q^2). \tag{3.1.14}$$

We can also consider the total cross sections in which the invariant dilepton mass Q^2 is integrated over. Γ_V is sufficiently small that we can use the *narrow width approximation*,

$$\frac{1}{(Q^2 - M_V^2)^2 + M_V^2 \Gamma_V^2} \simeq \frac{\pi}{M_V \Gamma_V} \delta(Q^2 - M_V^2). \tag{3.1.15}$$

Then integrating equation (3.1.9) and using

$$\sigma_V = \frac{\Gamma_V}{\Gamma_{V\to l_1\bar{l}_2}}\sigma_{V\to l_1\bar{l}_2}, \tag{3.1.16}$$

we find

$$\sigma_Z(\tau) = \frac{\pi G_F\sqrt{2}}{4N_c}\tau W_Z(\tau, M_Z^2) \tag{3.1.17}$$

$$\sigma_W(\tau) = \frac{\pi G_F\sqrt{2}}{N_c}\tau W_W(\tau, M_W^2). \tag{3.1.18}$$

These are the total cross sections. As equation (3.1.16) dictates, to obtain the cross sections for leptonic decay, $\sigma_{V\to l_1\bar{l}_2}$, we must multiply σ_Z by $B_Z = 0.03366$ and σ_W by $B_W = 0.108$, where B_V are the *branching ratios*, given by

$$B_V = \frac{\Gamma_{V\to l_1\bar{l}_2}}{\Gamma_V}. \tag{3.1.19}$$

Finally, note it is more common to talk about the centre of momentum energy, $\sqrt{s}$, instead of τ, where

$$s = \frac{Q^2}{\tau}. \tag{3.1.20}$$

3.2 The Parton Model In Drell-Yan

We now apply the parton model to Drell-Yan. In hadron h_α, the parton momenta for leading twist calculations is given by

$$p_\alpha = \xi_\alpha P_\alpha. \tag{3.2.1}$$

Defining

$$\tau' = \frac{q^2}{(p_1 + p_2)^2}, \tag{3.2.2}$$

then, since $P_\alpha^2 = 0$ for $\alpha = 1, 2$, we have

$$\tau' = \frac{\tau}{\xi_1 \xi_2} \tag{3.2.3}$$

Let the PDF of parton i in hadron h_α be written f_{0i}^α. We shall omit the "V" subscript on W_V from now on. $W(\tau, Q^2)$ is given by

$$W(\tau, Q^2) = \sum_{i,j} \int_\tau^1 \frac{d\xi_1}{\xi_1} \int_{\frac{\tau}{\xi_1}}^1 \frac{d\xi_2}{\xi_2} \widehat{W}_{ij} \left(\frac{\tau}{\xi_1 \xi_2}, \frac{Q^2}{\kappa^2}, a_s(Q^2) \right) f_{0i}^1(\xi_1) f_{0j}^2(\xi_2). \tag{3.2.4}$$

The sum is over partons of all types. The limits on ξ_α are found as follows. Since the hadronic state that results after the parton has left hadron α has momentum $(1 - \xi_\alpha)P_\alpha$, we have $\xi_\alpha \leq 1$. Since the final state of the parton-parton system is real, we must have

$$(\xi_1 P_1 + \xi_2 P_2)^2 \geq 0, \tag{3.2.5}$$

or

$$\xi_1 \xi_2 P_1 \cdot P_2 \geq 0. \tag{3.2.6}$$

But $P_1 \cdot P_2 \geq 0$; $P_1 \cdot P_2 \neq 0$ since the hadrons are moving relative to one another, so

$$\xi_1 \xi_2 \geq 0. \tag{3.2.7}$$

Thus the ξ_α must have the same sign. If they are both negative, the energy of the final state created by the initial parton-parton system, given by $\xi_1 P_1^0 + \xi_2 P_2^0$, is negative. Thus we must have

$$\xi_\alpha \geq 0. \tag{3.2.8}$$

It is possible to simplify equation (3.2.4) by taking the Mellin transform in the form

$$W(N,Q^2) = \int_0^1 d\tau \tau^{N-1} W(\tau,Q^2). \tag{3.2.9}$$

Rewriting equation (3.2.4) in the form

$$W(\tau,Q^2) = \sum_{i,j} \int_0^1 d\xi_1 \int_0^1 d\xi_2 \int_0^1 d\xi_3 \; \delta(\tau - \xi_1\xi_2\xi_3)$$
$$\times \widehat{W}_{ij}\left(\xi_3, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) f_{0i}^1(\xi_1) f_{0j}^2(\xi_2), \tag{3.2.10}$$

we find

$$W(N,Q^2) = \sum_{i,j} \widehat{W}_{ij}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) f_{0i}^1(N) f_{0j}^2(N). \tag{3.2.11}$$

The $\widehat{W}_{ij}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$ contain mass singularities. Defining a new quantity $\widetilde{W}_{ij}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right)$ such that

$$\widehat{W}_{kl}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$$
$$= \sum_{i,j} \Gamma_{ik}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right) \widetilde{W}_{ij}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right) \Gamma_{jl}\left(N, \frac{\mu_F^2}{\kappa^2}, a_s(\mu_F^2)\right), \tag{3.2.12}$$

it turns out from the factorization theorem for Drell-Yan [6] that the $\widetilde{W}_{ij}$ are free of mass singularities. Now, after rewriting the $f_{0j}^\alpha(N)$ in terms of $f_i^\alpha(N,\mu_F^2)$ using equation (2.8.3), and then rewriting the $\widehat{W}_{ij}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$ in terms of the factorized $\widetilde{W}_{ij}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right)$ using equation (3.2.12), equation (3.2.4) becomes

$$W(N,Q^2) = \sum_{i,j} \widetilde{W}_{ij}\left(N, \frac{Q^2}{\mu_F^2}, a_s(\mu_F^2)\right) f_i^1(N,\mu_F^2) f_j^2(N,\mu_F^2). \tag{3.2.13}$$

With $\mu_F = Q$,

$$W_Z(N,Q^2) = \sum_{I=1}^{n_f} K_I \Big[\widetilde{\Delta}_{q\bar{q}}(N, a_s(Q^2)) q_I^1(N,Q^2)\bar{q}_I^2(N,Q^2)$$

$$+\widetilde{\Delta}_{qg}(N, a_s(Q^2))(q_I^1(N,Q^2) + \bar{q}_I^1(N,Q^2)) g^2(N,Q^2)\Big] + 1 \leftrightarrow 2 \tag{3.2.14}$$

$$W_{W^+}(N,Q^2) = \sum_{J'',I' \le n_f} |V_{J''I'}|^2 \Big[\widetilde{\Delta}_{q\bar{q}}(N, a_s(Q^2)) q_{J''}^1(N,Q^2)\bar{q}_{I'}^2(N,Q^2)$$

$$+\widetilde{\Delta}_{qg}(N, a_s(Q^2))\left(q_{J''}^1(N,Q^2) + \bar{q}_{I'}^1(N,Q^2)\right) g^2(N,Q^2)\Big] + 1 \leftrightarrow 2, \tag{3.2.15}$$

where I' is odd and J'' is even, and where

$$K_I = 4e_{q_I}^2\left(1 - 2\sin^2\theta_W\right)^2 - \frac{4}{3}e_{q_I}\left(1 - 2\sin^2\theta_W\right) + \frac{10}{9}. \tag{3.2.16}$$

The $\widetilde{\Delta}_{ij}$ are the Drell-Yan *coefficient functions*. W_{W^-} is found by interchanging quarks and antiquarks in equation (3.2.15). From [46], to NLO,

$$\widetilde{\Delta}_{q\bar{q}}(\xi, a_s(Q^2)) = \delta(1-\xi) + a_s(Q^2)C_F\Big([4\zeta(2) - 8]\delta(1-\xi) + 8\left[\frac{\ln(1-\xi)}{1-\xi}\right]_+$$

$$-4(1+\xi)\ln(1-\xi) - 2\frac{1+\xi^2}{1-\xi}\ln\xi\Big) \tag{3.2.17}$$

$$\widetilde{\Delta}_{qg}(\xi, a_s(Q^2)) = a_s(Q^2)T_R n_f\Big((1 + 2\xi^2 - 2\xi)(2\ln(1-\xi) - \ln\xi)$$

$$+\frac{1}{2} - \frac{7}{2}\xi^2 + 3\xi\Big). \tag{3.2.18}$$

The meaning of the "+" in the $\left[\frac{\ln(1-\xi)}{1-\xi}\right]_+$ term of equation (3.2.17) is explained in section B.2. In N space, these become

$$\tilde{\Delta}_{q\bar{q}}(N, a_s(Q^2)) = 1 + a_s(Q^2)C_F\left[8\zeta(2) - 8 + 2(S_1^2(N-1) + S_1^2(N+1))\right] \tag{3.2.19}$$

$$\tilde{\Delta}_{qg}(N, a_s(Q^2)) = a_s(Q^2)T_R n_f\Big[4\frac{S_1(N+1)}{N+1} - 2\frac{S_1(N)}{N} - 4\frac{S_1(N+2)}{N+2} + \frac{1}{N^2} + \frac{2}{(N+2)^2} - \frac{2}{(N+1)^2} + \frac{1}{2N} - \frac{7}{2(N+2)} + \frac{3}{N+1}\Big]. \tag{3.2.20}$$

It is simple to extend the results of (2.18) to Drell-Yan. The scale dependence of the evolution of the PDF's is the same as before. For the Drell-Yan coefficient functions, we can use a similar derivation to that used to obtain equation (2.18.13). Omitting parton indices for brevity, we find to NLO that

$$\tilde{\Delta}\left(N, \frac{Q^2}{\mu_F^2}, a_s(Q^2)\right) = \tilde{\Delta}^{(0)}(N) + \left(\tilde{\Delta}^{(1)}(N) - 2\tilde{\Delta}^{(0)}(N)P^{(0)}(N)\ln k_F\right) a_s(k_R Q^2). \tag{3.2.21}$$

Chapter 4

Resummation Of Soft Singularities

Quark initiated bare cross sections contain divergences of the form $\ln^m N$ as $N \to \infty$, where N is the Mellin variable in DIS and Drell-Yan. These logarithms result from soft (and eventually collinear) gluon emission, and exponentiate [47, 48, 49]. Much progress has been made [47, 48] in resumming these soft singularities in the argument of the exponential, in the form shown in equation (4.1.8) below.

However, resummation of soft singularities is also necessary in the phenomenological *factorized* quark initiated cross sections, the quark coefficient functions, to improve the convergence of their perturbation series at large N. In [50], it has been proven that the $\overline{MS}$ quark splitting functions behave at large N like $O(\ln N)$ to all orders, so that these splitting functions are in fact resummed at large N. The $\overline{MS}$ quark coefficient functions resummed at large N are obtained using this result [51, 52]. However, it is not clear whether $\overline{MS}$ factorization is being correctly applied, since the $\overline{MS}$ scheme is only defined when dimensional regularization is used to regulate the infrared singularities in bare cross sections, whereas [47] uses an infrared cutoff, κ. In this chapter we shall formally derive the form of the $\overline{MS}$ quark coefficient functions for DIS and Drell-Yan that resums all soft singularities, and give an alternative proof that the $\overline{MS}$ quark splitting functions behave at large N like $O(\ln N)$ to all orders, by obtaining the form of

the splitting functions that resums all soft singularities.

4.1 Large N Resummation Of Bare Quark Initiated Cross Sections

Some of the following has been published in [53]. See appendix B.2 for a discussion of the "plus prescription", which will be used in this section. In DIS and Drell-Yan bare quark initiated cross sections, when the initial quark (or two quarks in the case of Drell-Yan) is (are each) connected to the electroweak boson via a single quark line, i.e. the diagrams contributing to a bare non singlet cross section, we find soft gluons radiated from this quark line, which occur as the scaling variable $z \to 1$. These gluons generate large $\left[\frac{\ln^i(1-z)}{1-z}\right]_+$ terms in the perturbation series, which need to be resummed to improve the convergence of the perturbation series. In N space, these logarithms become

$$\int_0^1 dz z^{N-1} \left[\frac{\ln^n(1-z)}{1-z}\right]_+ = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \ln^n(1-z)$$
$$= \sum_{i=0}^{n+1} b_i^n \ln^i N + O\left(\frac{1}{N}\right). \tag{4.1.1}$$

Note that we do not distinguish terms of $O\left(\frac{\ln^m N}{N}\right)$ from terms of $O\left(\frac{1}{N}\right)$. Thus we see that singularities as $z \to 1$ manifest themselves in N space as singularities as $N \to \infty$. In z space there are also terms proportional to $\delta(1-z)$ in a bare non singlet quark initiated cross section, which are not generated solely by soft gluon radiation, which in N space are constant as $N \to \infty$ since

$$\int_0^1 dz z^{N-1} \delta(1-z) = 1. \tag{4.1.2}$$

All remaining terms in a bare non singlet quark initiated cross section, and all diagrams in a bare quark initiated cross section in which the initial quark (or at least one of the two initial quarks in the case of Drell-Yan) is not connected to the electroweak boson via a single quark line, i.e. the diagrams contributing to a

bare pure singlet cross section, are either non singular or of the form $\ln^m(1-z)$ as $z \to 1$.These remaining non singlet terms and pure singlet diagrams in N space are then of $O\left(\frac{1}{N}\right)$, since for any non singular function $f(z)$,

$$\left| \int_0^1 dz z^{N-1} f(z) \right| \leq \frac{1}{N} \max|f(z)|, \tag{4.1.3}$$

and the Mellin transform of $\ln^m(1-z)$ behaves like

$$\lim_{N\to\infty} \int_0^1 dz z^{N-1} \ln^m(1-z) = O\left(\frac{1}{N}\right). \tag{4.1.4}$$

Let $\widehat{W}_{V\mu\nu}^{[1]q_I NS}$ denote the non singlet quark initiated parts of the DIS tensor, and $\widehat{W}_{V}^{[2]q_I\bar{q}_J NS}$ denote the non singlet quark initiated part of the Drell-Yan cross section, where V labels the electroweak process in both cases. Now soft gluon radiation does not depend on the quantum numbers of the electroweak boson involved. Consequently in N space, the $O(\ln^m N)$ terms for $m = 1, ..., \infty$ are independent of V, μ and ν, up to electroweak factors. (Note that μ and ν only describe the polarization of the electroweak bosons either side of the cut in the cut diagram.) Furthermore, terms of $O(1)$ are independent of V, μ and ν, since these terms contain contributions only from soft gluons and from terms proportional to $\delta(1-z)$ in z space, where the latter arise from momentum conservation of the renormalized incoming quark line connected to a QCD renormalized electroweak boson-quark-quark vertex, which is independent of the quantum numbers of the electroweak boson. Consequently, in DIS for example, the $O(\ln^m N)$ terms for $m = 0, ..., \infty$ in $\widehat{F}_{1I}$, $\widehat{F}_{2I}$ and $\widehat{F}_{3I}$ are identical, up to electroweak factors. This is because these three quantities are linear combinations of the components of $W_{\mu\nu}$, where the weighting factors are given in equations (1.2.19), (1.2.20) and (1.2.22). Thus without loss of generality we may write

$$\widehat{W}_{V\mu\nu}^{[1]q_I NS} = \widehat{W}_{V\mu\nu}^{[1]\bar{q}_I} + O\left(\frac{1}{N}\right) = K_{\mu\nu}^{[1]V} \hat{\sigma}_q^{[1]} + O\left(\frac{1}{N}\right) \tag{4.1.5}$$

$$\widehat{W}_{V}^{[2]q_I\bar{q}_J NS} = \widehat{W}_{V}^{[2]\bar{q}_I q_J} + O\left(\frac{1}{N}\right) = K^{[2]V} \hat{\sigma}_q^{[2]} + O\left(\frac{1}{N}\right), \tag{4.1.6}$$

where the $K_{\mu\nu}^{[1]V}$ and $K^{[2]V}$ contain all electroweak factors, which we are not interested in. Note that in $\widehat{\sigma}_q^{[a]}$, a refers to the number of initial quarks. By these definitions, we are free to define the $O\left(\frac{1}{N}\right)$ parts of the $\widehat{\sigma}_q^{[a]}$, which we will do later. In general [54], the perturbation series for $\widehat{\sigma}_q^{[a]}$ in N space is of the form

$$\widehat{\sigma}_q^{[a]}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right)$$

$$= 1 + \sum_{n=1}^{\infty} a_s^n(Q^2) \sum_{m=0}^{2n} C_{nm}^{[a]} \left(\ln \frac{Q^2}{\kappa^2}\right) \ln^m N + O\left(\frac{1}{N}\right), \tag{4.1.7}$$

where $\widehat{\sigma}_q^{[a]}$ is normalized such that $\widehat{\sigma}_q^{[a]} = 1$ for $a_s = 0$. κ regulates the mass divergences. Terms of the form $n+1 \leq m \leq 2n$ are called *leading logarithms* (LL) and terms of the form $m = n$ are called *next to leading logarithms* (NLL). All other logarithms are referred to as *sub-dominant.*

In [55], it is shown that a) all the $O(\ln^m N)$ terms, for $m = 0, ..., \infty$, factor out of bare quark initiated cross sections and b) all these terms exponentiate. Then, it turns out that these singularities can be *resummed* in the exponent. Resummation is the process of reorganising the all orders perturbation series such that in the new perturbation series the singularities cancel order by order. From [49], after resummation the cross section takes the form

$$\ln \widehat{\sigma}_q^{[a]}\left(N, \frac{Q^2}{\kappa^2}, a_s(Q^2)\right) = g_{-1}^{[a]}\left(a_s(Q^2) \ln N, \frac{Q^2}{\kappa^2}\right) \ln N$$

$$+ \sum_{n=0}^{\infty} a_s^n(Q^2) g_n^{[a]}\left(a_s(Q^2) \ln N, \frac{Q^2}{\kappa^2}\right) + O\left(\frac{1}{N}\right), \tag{4.1.8}$$

This resummation is achievable essentially by choosing $\mu_R^2 = (1-z)^a Q^2$ in the perturbation series in z space, i.e. by expanding in a coupling $a_s((1-z)^a Q^2)$, where a is an integer, instead of $a_s(Q^2)$; it turns out that this choice removes all large N singularities in the coefficients of a_s^n in the new series, for all n. In [56], the same exponentiation followed by resummation as above is obtained directly by application of the renormalization group and gauge invariance to the factorization mentioned in a) above. Since at present calculations in DIS and Drell-Yan are only to NLO, one is only interested in g_{-1}, which contains all the LL terms, and

g_0, which contains all the NLL terms, since these are the only types of logarithms of N that appear in the usual unresummed NLO expressions.

The results in [47, 54] read

$$\ln \widehat{\sigma}_q^{[1]}\left(N-1,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)=\int_0^1 dz\frac{z^{N-1}-1}{1-z}\Big(B(a_s((1-z)Q^2))+\int_{\kappa^2}^{(1-z)Q^2}\frac{dq^2}{q^2}A(a_s(q^2))\Big)+O(1) \tag{4.1.9}$$

$$\ln \widehat{\sigma}_q^{[2]}\left(N-1,\frac{Q^2}{\kappa^2},a_s(Q^2)\right)=\int_0^1 dz\frac{z^{N-1}-1}{1-z}\int_{\kappa^2}^{(1-z)^2Q^2}\frac{dq^2}{q^2}A(a_s(q^2))+O(1). \tag{4.1.10}$$

where the $\overline{MS}$ scheme has been used for renormalization. Writing

$$A(a_s)=\sum_{i=1}^{\infty}A^{(i)}a_s^i \qquad B(a_s)=\sum_{i=1}^{\infty}B^{(i)}a_s^i, \tag{4.1.11}$$

then the coefficients that need to be known if we wish to resum LL and NLL terms are $A^{(1)}$, $A^{(2)}$ and $B^{(1)}$. These have been found to be

$$A^{(1)}=2C_F \qquad A^{(2)}=\left[\frac{67}{9}-\frac{\pi^2}{3}\right]C_AC_F-\frac{20}{9}T_Rn_fC_F \qquad B^{(1)}=-\frac{3}{2}C_F. \tag{4.1.12}$$

We now go to $d=4-2\epsilon$ dimensions. Then we may take $\kappa\to 0$, since bare cross sections are non singular in a non integer number of dimensions. Then the DIS cross section takes the form

$$\ln \widehat{\sigma}_q^{[1]}=\int_0^1 dz\frac{z^{N-1}-1}{1-z}\Big(B(a_s((1-z)Q^2,\epsilon),\epsilon)+\int_0^{(1-z)Q^2}\frac{dq^2}{q^2}A(a_s(q^2,\epsilon),\epsilon)\Big)+r_1(N,Q^2,\epsilon)+O(\epsilon), \tag{4.1.13}$$

and the Drell-Yan cross section takes the form

$$\ln \widehat{\sigma}_q^{[2]} = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \int_0^{(1-z)^2 Q^2} \frac{dq^2}{q^2} A(a_s(q^2,\epsilon),\epsilon) + r_2(N,Q^2,\epsilon) + O(\epsilon). \tag{4.1.14}$$

All necessary details on dimensional regularization and the $\overline{MS}$ scheme in renormalization are given in appendix C.1. Note that $a_s(q^2,\epsilon)$ is the dimensionally continued version of $a_s(q^2)$, defined by equation (C.1.4), and it obeys $a_s(q^2,0) = a_s(q^2)$. $A(a_s,\epsilon)$ and $B(a_s,\epsilon)$ have the expansions

$$A(a_s,\epsilon) = \sum_{i=1}^{\infty} A^{(i)}(\epsilon) a_s^i \qquad B(a_s,\epsilon) = \sum_{i=1}^{\infty} B^{(i)}(\epsilon) a_s^i. \tag{4.1.15}$$

The $r_a(N,Q^2,\epsilon)$ are singular as $\epsilon \to 0$, but are finite in the limit $N \to \infty$. They contain contributions from processes not only related to soft gluons. It will turn out that the extra terms of $O(\epsilon)$ in equations (4.1.13) and (4.1.14) will not affect our later work. Using equation (C.1.8), we do the q^2 integrals in the form

$$\int_0^{Q^2} \frac{dq^2}{q^2} a_s^n(q^2,\epsilon) = \int_0^{a_s(Q^2,\epsilon)} da_s \frac{a_s^n}{-\epsilon a_s - \sum_{j=0}^{\infty} \beta_j a_s^{j+2}}. \tag{4.1.16}$$

Then from equations (4.1.13) and (4.1.14), we find the general form for quark initiated cross sections at large N to be

$$\ln \widehat{\sigma}_q^{[a]}(N, a_s(Q^2,\epsilon),\epsilon) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \left[\sum_{i=1}^{\infty} f^{[a](i)}(\epsilon) a_s^i((1-z)^a Q^2,\epsilon) \right] + r_a(N,Q^2,\epsilon) + O(\epsilon). \tag{4.1.17}$$

We write the $r_a(N,Q^2,\epsilon)$ in the form

$$r_a(N,Q^2,\epsilon) = \sum_{i=1}^{\infty} f_\delta^{[a](i)}(\epsilon) a_s^i(Q^2,\epsilon) + O\left(\frac{1}{N}\right). \tag{4.1.18}$$

Since we are free to define the terms of $O\left(\frac{1}{N}\right)$ in $\widehat{\sigma}_q^{[a]}$, we may *define* $\widehat{\sigma}_q^{[a]}$ to be

$$\ln \widehat{\sigma}_q^{[a]}(N, a_s(Q^2,\epsilon),\epsilon) = \int_0^1 dz \frac{z^{N-1}-1}{1-z} \left[\sum_{i=1}^{\infty} f^{[a](i)}(\epsilon) a_s^i((1-z)^a Q^2,\epsilon) \right]$$

$$+\sum_{i=1}^{\infty} f_{\delta}^{[a](i)}(\epsilon) a_s^i(Q^2,\epsilon) + O(\epsilon). \tag{4.1.19}$$

Note that the term of $O(\epsilon)$ in equation (4.1.19) is not necessarily of $O\left(\frac{1}{N}\right)$.

4.2 Large N Resummation Of Factorized Quark Coefficient Functions And Splitting Functions

We now factorize equation (4.1.19) in the $\overline{MS}$ scheme. All necessary details on dimensional regularization and the $\overline{MS}$ scheme in factorization are given in appendix C.2.

Consider DIS. We place the index "[1]" on the DIS coefficient functions to distinguish them from Drell-Yan coefficient functions, which will have the index "[2]". Recall the form the $\widehat{F}_{1,2,3I}$ take, equation (2.13.1). We are not concerned with the electroweak factors $K_{\Theta I}^V$. Since the bare pure singlet coefficient functions $\hat{C}_{\Theta qS}^{[1]} - \hat{C}_{\Theta qNS}^{[1]}$ behave like $O\left(\frac{1}{N}\right)$ at large N, all the $O(\ln^m N)$ terms for $m = 0, ..., \infty$ of $\hat{C}_{\Theta qS}^{[1]}$ are the same as those in $\hat{C}_{\Theta qNS}^{[1]}$, up to electroweak factors. Furthermore, from equation (4.1.5), the $O(\ln^m N)$ terms for $m = 0, ..., \infty$ of $\widehat{F}_{\Theta I}$ are the same, up to electroweak factors. Thus the five bare coefficient functions $\hat{C}_{1,2qNS}^{[1]}$, $\hat{C}_{1,2qS}^{[1]}$ and $\hat{C}_{3qNS}^{[1]}$ are identical up to terms of $O\left(\frac{1}{N}\right)$. Likewise the $O(\ln^m N)$ terms for $m = 0, ..., \infty$ of the Drell-Yan bare non singlet coefficient function $\hat{C}_{qNS}^{[2]}$ are identical to those in $\hat{C}_{qS}^{[2]}$. Thus, from equations (4.1.5) and (4.1.6), omitting the Θ subscript on the DIS coefficient functions with no loss of generality,

$$\ln \hat{C}_{qNS}^{[a]} = \ln \widehat{\sigma}_q^{[a]} + O\left(\frac{1}{N}\right) \tag{4.2.1}$$

$$\ln \hat{C}_{qS}^{[a]} = \ln \widehat{\sigma}_q^{[a]} + O\left(\frac{1}{N}\right) \tag{4.2.2}$$

The factorization procedure in the non singlet case can be written as

$$\ln C_{qNS}^{[a]} = \ln \hat{C}_{qNS}^{[a]} - a \ln \Gamma_{(+)}, \tag{4.2.3}$$

where the $C_{qNS}^{[a]}$ are the non singlet coefficient functions, which are non singular as $\epsilon \to 0$. We first use equation (4.2.3) to determine the degree of divergence of the $f^{[a](i)}(\epsilon)$ and $f_\delta^{[a](i)}(\epsilon)$ as $\epsilon \to 0$. Differentiating equation (4.2.3) with respect to $\ln Q^2$, and using equation (C.2.17), we find

$$\frac{d \ln \hat{C}_{qNS}^{[a]}}{d \ln Q^2} = \frac{d \ln C_{qNS}^{[a]}}{d \ln Q^2} + a a_s(Q^2) \sum_{n=0}^{\infty} P_{(+)}^{(n)} a_s^n(Q^2). \tag{4.2.4}$$

Thus $\frac{d \ln \hat{C}_{qNS}^{[a]}}{d \ln Q^2}$ is non singular as $\epsilon \to 0$. This leads to the results

$$f^{[a](i)}(\epsilon) = \sum_{j=0}^{i} f^{[a](i),j} \epsilon^{-j} \qquad f_\delta^{[a](i)}(\epsilon) = \sum_{j=0}^{i} f_\delta^{[a](i),j} \epsilon^{-j}, \tag{4.2.5}$$

as well as relations among the $f^{[a](i),j}$, and among the $f_\delta^{[a](i),j}$, which eliminates some of them. The remaining ones can be obtained by expanding equation (4.1.19) as a series in $a_s(Q^2)$ and comparing with the usual unresummed cross sections. We have not included terms of $O(\epsilon)$ or higher in the definitions of the $f^{[a](i)}(\epsilon)$ and $f_\delta^{[a](i)}(\epsilon)$, since these terms can be absorbed into the $O(\epsilon)$ part of equation (4.1.19).

Using equation (C.2.18), we can rewrite $\ln \Gamma_{(+)}$ in a form more suitable for our purposes:

$$\ln \Gamma_{(+)}(N, a_s(Q^2, \epsilon), \epsilon) = \int_0^1 dz z^{N-1} \int_0^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}(z, a_s(\lambda Q^2, \epsilon)). \tag{4.2.6}$$

Note that equation (4.2.6) defines the $\overline{MS}$ scheme: the $P_{(+)}^{(n)}$ are independent of ϵ. Let us write

$$P_{(+)} = P_{(+)}^{\sigma_q} + P_{(+)}^{\text{rem}} \tag{4.2.7}$$

where $P_{(+)}^{\sigma_q}$ is defined such that

$$\ln \widetilde{\sigma}_q^{[a]} = \ln \widehat{\sigma}_q^{[a]} \doteq a \int_0^1 dz z^{N-1} \int_0^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)) \tag{4.2.8}$$

is non singular as $\epsilon \to 0$. Note that the remainder term

$$P_{(+)}^{\rm rem}(N, a_s) = O\left(\frac{1}{N}\right), \tag{4.2.9}$$

as this quantity is used to subtract the $\epsilon \to 0$ singularities in the $O\left(\frac{1}{N}\right)$ term of equation (4.2.1). Thus equation (4.2.3) can be written

$$\begin{aligned}
\ln C_{qNS}^{[a]} = & \int_0^1 dz (z^{N-1} - 1) \Bigg[\frac{1}{1-z} \sum_{i=1}^{\infty} \sum_{t=1}^{i} f^{[a](i),t} \epsilon^{-t} a_s^i((1-z)^a Q^2, \epsilon) \\
& - a \int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)) \Bigg] \\
& + \left[\sum_{i=1}^{\infty} \sum_{t=1}^{i} f_\delta^{[a](i),t} \epsilon^{-t} a_s^i(Q^2, \epsilon) - a \int_0^1 dz \int_0^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)) \right] \\
& + \Bigg[\int_0^1 dz (z^{N-1} - 1) \frac{1}{1-z} \sum_{i=1}^{\infty} f^{[a](i),0} a_s^i((1-z)^a Q^2, \epsilon) + \sum_{i=1}^{\infty} f_\delta^{[a](i),0} a_s^i(Q^2, \epsilon) \\
& - a \int_0^1 dz (z^{N-1} - 1) \int_{(1-z)^a}^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)) \Bigg] + O\left(\frac{1}{N}\right) + O(\epsilon).
\end{aligned} \tag{4.2.10}$$

The motivation for equation (4.2.10) is that, as we will see, each of the three pairs of square brackets is finite as $\epsilon \to 0$. We first work with the first pair of square brackets. Using equation (C.1.8), we can perform the integration of the $a_s^n(\lambda Q^2)$ in the second term in the first pair of square brackets as

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s^n(\lambda Q^2, \epsilon) = \int_0^{a_s((1-z)^a Q^2, \epsilon)} da_s \frac{a_s^n}{-\epsilon a_s - \sum_{j=0}^{\infty} \beta_j a_s^{j+2}}. \tag{4.2.11}$$

This gives the form

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s^n(\lambda Q^2, \epsilon) = \sum_{i=n}^{\infty} p^{(i),n}(\epsilon) a_s^i((1-z)^a Q^2, \epsilon). \tag{4.2.12}$$

The second term in the first pair of square brackets in equation (4.2.10) can therefore be written

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)) = \sum_{n=1}^{\infty} \int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s^n(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q(n-1)}(z)$$

$$= \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} P_{(+)}^{\sigma_q(n-1)}(z) p^{(i),n}(\epsilon) a_s^i((1-z)^a Q^2, \epsilon). \tag{4.2.13}$$

Reorganizing the sums gives

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon))$$

$$= \sum_{i=1}^{\infty} \sum_{n=1}^{i} P_{(+)}^{\sigma_q(n-1)}(z) p^{(i),n}(\epsilon) a_s^i((1-z)^a Q^2, \epsilon). \tag{4.2.14}$$

We now expand equation (4.2.14) in ϵ. In equation (4.2.12), the $p^{(i),n}(\epsilon)$ obey

$$p^{(i),n}(\epsilon) = \sum_{d=1}^{i-n+1} p_d^{(i),n} \epsilon^{-d}. \tag{4.2.15}$$

Note that $p_d^{(i),n} = 0$ if $d = 1$ and $i \neq n$, but we do not explicitly show this in order to keep our notation simpler. The $p_d^{(i),n}$ can be obtained from equation (4.2.11). For example, the first few terms are

$$p_1^{(1),1} = -1 \qquad p_2^{(2),1} = \frac{\beta_0}{2} \qquad p_1^{(2),2} = -\frac{1}{2}. \tag{4.2.16}$$

Thus with the expansion in equation (4.2.15), equation (4.2.14) becomes

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon))$$

$$= \sum_{i=1}^{\infty} \sum_{n=1}^{i} \sum_{d=1}^{i-n+1} P_{(+)}^{\sigma_q(n-1)}(z) p_d^{(i),n} \epsilon^{-d} a_s^i((1-z)^a Q^2, \epsilon). \tag{4.2.17}$$

Reorganizing the sums gives

$$\int_0^{(1-z)^a} \frac{d\lambda}{\lambda} a_s(\lambda Q^2,\epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2,\epsilon))$$

$$= \sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n} \epsilon^{-t} a_s^i((1-z)^a Q^2,\epsilon). \qquad (4.2.18)$$

For the second term in the second pair of square brackets in equation (4.2.10), we find by setting $a = 0$ in equation (4.2.18) that

$$\int_0^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2,\epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2,\epsilon)) = \sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n} \epsilon^{-t} a_s^i(Q^2,\epsilon). \qquad (4.2.19)$$

Finally, we find that the last term in the third pair of brackets in equation (4.2.10) is finite as $\epsilon \to 0$. Taking this limit, we can write

$$\int_{(1-z)^a}^1 \frac{d\lambda}{\lambda} a_s(\lambda Q^2,\epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2,\epsilon)) = \int_{a_s((1-z)^a Q^2)}^{a_s(Q^2)} da_s \sum_{n=1}^{\infty} \frac{a_s^n P_{(+)}^{\sigma_q(n-1)}(z)}{\sum_{j=0}^{\infty} \beta_j a_s^{j+2}}. \qquad (4.2.20)$$

Thus equation (4.2.10) now reads

$$\ln C_{qNS}^{[a]} = \int_0^1 dz (z^{N-1}-1)\Bigg[\frac{1}{1-z}\sum_{i=1}^{\infty}\sum_{t=1}^{i} f^{[a](i),t}\epsilon^{-t} a_s^i((1-z)^a Q^2,\epsilon)$$

$$- a\sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n}\epsilon^{-t} a_s^i((1-z)^a Q^2,\epsilon)\Bigg]$$

$$+\Bigg[\sum_{i=1}^{\infty}\sum_{t=1}^{i} f_\delta^{[a](i),t}\epsilon^{-t} a_s^i(Q^2,\epsilon) - a\int_0^1 dz \sum_{i=1}^{\infty}\sum_{t=1}^{i}\sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n}\epsilon^{-t} a_s^i(Q^2,\epsilon)\Bigg]$$

$$+\Bigg[\int_0^1 dz (z^{N-1}-1)\frac{1}{1-z}\sum_{i=1}^{\infty} f^{[a](i),0} a_s^i((1-z)^a Q^2,\epsilon) + \sum_{i=1}^{\infty} f_\delta^{[a](i),0} a_s^i(Q^2,\epsilon)$$

$$- a\int_0^1 dz (z^{N-1}-1)\int_{a_s((1-z)^a Q^2)}^{a_s(Q^2)} da_s \sum_{n=1}^{\infty}\frac{a_s^n P_{(+)}^{\sigma_q(n-1)}(z)}{\sum_{j=0}^{\infty}\beta_j a_s^{j+2}}\Bigg] + O\left(\frac{1}{N}\right) + O(\epsilon). \qquad (4.2.21)$$

There are three square bracketed terms in equation (4.2.21), hereafter referred to as terms 1, 2 and 3, respective of the order of appearance in equation (4.2.21). Consider only the coefficients of ϵ^{-t} for $t = 1, ..., \infty$ in equation (4.2.21), which must vanish so that $\ln C_{qNS}^{[a]}$ is finite in the limit $\epsilon \to 0$. In term 1, these coefficients are all functions of N. In term 2, these coefficients are all independent of N. All subterms in term 3 are non singular as $\epsilon \to 0$. Subtracting term 2 and term 3 from both sides of the equation, and then differentiating both sides of the equation with respect to N, we find that term 1 is non singular as $\epsilon \to 0$, and therefore so is term 2. In conclusion, terms 1, 2 and 3 are each non singular as $\epsilon \to 0$. When term 1 has been differentiated with respect to N, the cancellation of singularities as $\epsilon \to 0$, as well as the fact that the Mellin transform is invertible, implies that for $i \geq t \geq 1$,

$$(\ln z)\frac{1}{1-z} f^{[a](i),t} = (\ln z) a \sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n}, \tag{4.2.22}$$

so that, for $z \neq 1$,

$$\frac{1}{1-z} f^{[a](i),t} = a \sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n}. \tag{4.2.23}$$

If we treat $p_t^{(i),n}$ in equation (4.2.23) as a matrix with indices i and n, then this matrix is invertible since it is triangular. Then multiplication of both sides of equation (4.2.23) by the inverse of this matrix gives an expression with just $P_{(+)}^{\sigma_q(n-1)}$ on the right hand side, with the left hand side proportional to $\left(\frac{1}{1-z}\right)$. Thus we find that

$$P_{(+)}^{\sigma_q}(z, a_s(Q^2)) = \tilde{P}_{(+)}^{\sigma_q}(a_s(Q^2)) \frac{1}{1-z} \text{ for } z \neq 1. \tag{4.2.24}$$

For all z, we can therefore write

$$P_{(+)}^{\sigma_q}(z, a_s(Q^2)) = \tilde{P}_{(+)}^{\sigma_q}(a_s(Q^2)) \left[\frac{1}{1-z}\right]_+ + \bar{P}_{(+)}^{\sigma_q}(a_s(Q^2)) \delta(1-z). \tag{4.2.25}$$

Thus we have found that, in N space, as $N \to \infty$,

$$P_{(+)} = P_{qq}^{(V)} + P_{q\bar{q}}^{(V)} = O(\ln N). \tag{4.2.26}$$

The cancellation of singularities in term 2 in equation (4.2.21) as $\epsilon \to 0$ gives

$$f_\delta^{[a](i),t} = \int_0^1 dz \sum_{n=1}^{i-t+1} P_{(+)}^{\sigma_q(n-1)}(z) p_t^{(i),n} = \sum_{n=1}^{i-t+1} \bar{P}_{(+)}^{\sigma_q(n-1)} p_t^{(i),n}, \tag{4.2.27}$$

where we have used the result

$$\int_0^1 dz \left[\frac{1}{1-z}\right]_+ = 0. \tag{4.2.28}$$

Again, equation (4.2.27) can be rearranged to obtain all the $\bar{P}_{(+)}^{\sigma_q(n-1)}$.

Applying exactly the same reasoning to cross sections that couple only to valence quark PDF's, e.g. $\widehat{F}_{3I}$ for charged current scattering in DIS, we would find that, in N space,

$$P_{(-)} = P_{qq}^{(V)} - P_{q\bar{q}}^{(V)} = O(\ln N). \tag{4.2.29}$$

Furthermore, since the $f^{[a](i)}(\epsilon)$ and $f_\delta^{[a](i)}(\epsilon)$ for cross sections that couple only to valence quarks are the same as those for cross sections that couple only to non singlet quarks, we would find that $P_{(-)}^{\sigma_q} = P_{(+)}^{\sigma_q}$ exactly, so that

$$P_{qq}^{(V)} + P_{q\bar{q}}^{(V)} = P_{qq}^{(V)} - P_{q\bar{q}}^{(V)} + O\left(\frac{1}{N}\right). \tag{4.2.30}$$

Therefore from equations (4.2.26), (4.2.29) and (4.2.30), we conclude finally that

$$P_{qq}^{(V)} = O(\ln N) \qquad P_{q\bar{q}}^{(V)} = O\left(\frac{1}{N}\right). \tag{4.2.31}$$

Equation (4.2.31) implies that the evolution kernel for non singlet and valence quarks is also resummed at large N. For example, the evolution of valence quark PDF's, $f_I^-(N,\mu_F^2)$, where μ_F^2 is the factorization scale, takes the form

$$f_I^-(N,\mu_F^2) = E^-(N,\mu_F^2,\mu_{F0}^2) f_I^-(N,\mu_{F0}^2), \tag{4.2.32}$$

where from equation (2.9.9) the valence quark evolution kernel E^- is given by

$$E^-(N,\mu_F^2,\mu_{F0}^2) = \exp\left[\int_{a_s(\mu_{F0}^2)}^{a_s(\mu_F^2)} da_s \frac{a_s P_{(-)}(N,a_s)}{\beta(a_s)}\right], \tag{4.2.33}$$

and due to equation (4.2.29) the exponent is resummed at large N in the form of equation (4.1.8). The evolution of non singlet quark PDF's is the same, but with $P_{(-)} \to P_{(+)}$. Taking the limit $\epsilon \to 0$, equation (4.2.21) now reads

$$\ln C_{qNS}^{[a]} = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\sum_{i=1}^{\infty} f^{[a](i),0} a_s^i((1-z)^a Q^2)\right.$$
$$\left. + a\int_{a_s(Q^2)}^{a_s((1-z)^a Q^2)} da_s \sum_{n=1}^{\infty} \frac{a_s^n \tilde{P}_{qq}^{(V)(n-1)}}{\sum_{j=0}^{\infty}\beta_j a_s^{j+2}}\right] + \sum_{i=1}^{\infty} f_\delta^{[a](i),0} a_s^i(Q^2) + O\left(\frac{1}{N}\right). \tag{4.2.34}$$

Equation (4.2.34) is the general resummed form of the $\overline{MS}$ coefficient functions $C_{qNS}^{[a]}$ and $C_{qS}^{[a]}$. $a_s((1-z)^a Q^2)$ in equation (4.2.34) must be interpreted in the form

$$a_s((1-z)^a Q^2) = \sum_{i=1}^{\infty} a_s^i(Q^2) h^{(i)}(1 + a\ a_s(Q^2)\ln(1-z)), \tag{4.2.35}$$

so that

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} a_s^k((1-z)^a Q^2) = \sum_{i=k-1}^{\infty} a_s^i(Q^2) R^{[a](i)}(1 - a a_s(Q^2)\ln N). \tag{4.2.36}$$

Then, in equation (4.2.34), if we include all the terms up to $O(a_s^{i+1})$ in the z integral and all the terms up to $O(a_s^i)$ outside the z integral, we would resum all terms of the form $a_s^n(a_s \ln N)^m$ for $n = -1, ..., i$ in the coefficient function.

To calculate the $f^{[a](i),0}$ and the $f_\delta^{[a](i),0}$, we simply expand equation (4.2.34) in $a_s(Q^2)$, i.e. undo the resummation, and compare coefficients of $a_s^i(Q^2)$ with the usual unresummed expression. This will be done for LL and NLL terms in section 4.4.

We now make some heuristic arguments as to the large N behaviour of the rest of the perturbation series that make up the physical cross sections. To $O\left(\frac{1}{N}\right)$, $\hat{C}_{qS}^{[a]}$ does not mix with $\hat{C}_g^{[a]}$ under factorization [55], and so the factorization for the quark singlet takes the form of equation (4.2.3). Thus

$$\ln C_{qS}^{[a]} = \ln C_{qNS}^{[a]} + O\left(\frac{1}{N}\right). \tag{4.2.37}$$

The factorization of $\hat{C}_{qS}^{[a]}$ just discussed allows us to infer that $P_{FF} = P_{qq}^{(V)} + O\left(\frac{1}{N}\right)$, and so

$$P_{qq}^{(S)} + P_{q\bar{q}}^{(S)} = O\left(\frac{1}{N}\right). \tag{4.2.38}$$

Up to NNLO, it has been found that $C_g^{[a]} = O\left(\frac{1}{N}\right)$ [35, 36, 57, 46], and we shall assume that this behaviour continues to all orders.

The large N behaviour of P_{gg}, P_{qg} and P_{gq} have also been obtained in [50], and it has been found that

$$P_{gg} = O(\ln N) \qquad P_{qg} = \left(\frac{1}{N}\right) \qquad P_{gq} = \left(\frac{1}{N}\right). \tag{4.2.39}$$

Thus the singlet splitting functions are resummed.

4.3 The Landau Pole

The resummed coefficient functions for DIS and Drell-Yan each have a *Landau pole* at

$$a_s(Q^2)\beta_0 \ln N = 1 \tag{4.3.40}$$

$$2a_s(Q^2)\beta_0 \ln N = 1 \tag{4.3.41}$$

respectively. To get back the z-space formula, we will choose the contour of integration to lie to the left of the Landau pole, but still to the right of all other poles, for the following reasons: The coefficients of the poles in N space for $\mathrm{Re}(N) \leq 1$ and $\mathrm{Im}(N) = 0$ (not including the Landau pole, which occurs at $N = \exp\left[\frac{1}{a_s(Q^2)\beta_0}\right]$ for DIS and $N = \exp\left[\frac{1}{2a_s(Q^2)\beta_0}\right]$ for Drell-Yan) are the quantities that we want to calculate, since they are all we need to know in order to calculate the cross section in question at leading twist. The Landau pole, on the other hand, contributes only at higher twist. Thus at leading twist it does not matter whether we include it or not, as long as we stick to one prescription. This prescription is more convenient for numerics, and is called the *minimal prescription* [58].

Note that the Landau pole in z space, although of higher twist, can be singular if the functional dependence of $a_s(\mu_R^2)$ on its argument μ_R^2 is not chosen carefully for small μ_R^2. Working in Mellin space and using the minimal prescription removes the Landau pole in z space very simply.

4.4 Resummation of LL and NLL Terms

We now use equation (4.2.34) to resum LL and NLL terms. To NLO, equation (4.2.34) reads

$$\ln \tilde{\sigma}_q^{[a]}(N, a_s(Q^2)) = \int_0^1 dz \frac{z^{N-1}-1}{1-z}\Bigg[-\frac{a\tilde{P}^{(0)}}{\beta_0} \ln\left(\frac{a_s((1-z)^a Q^2)}{a_s(Q^2)}\right)$$
$$+\left(\frac{a\tilde{P}^{(0)}\beta_1}{\beta_0^2} - \frac{a\tilde{P}^{(1)}}{\beta_0}\right)\left(a_s((1-z)^a Q^2) - a_s(Q^2)\right) + f^{[a](1),0} a_s((1-z)^a Q^2)\Bigg]$$
$$+O(a_s(a_s \ln N)^m). \tag{4.4.1}$$

Now

$$a_s(\nu Q^2) = \frac{a_s(Q^2)}{1 + a_s(Q^2)\beta_0 \ln \nu} - a_s^2(Q^2)\frac{\beta_1}{\beta_0}\frac{\ln\left(1 + a_s(Q^2)\beta_0 \ln \nu\right)}{\left(1 + a_s(Q^2)\beta_0 \ln \nu\right)^2}$$
$$+O(a_s^3(a_s \ln \nu)^n), \tag{4.4.2}$$

and so

$$\ln \frac{a_s(\nu Q^2)}{a_s(Q^2)} = -\ln(1 + a_s(Q^2)\beta_0 \ln \nu) - a_s(Q^2)\frac{\beta_1}{\beta_0}\frac{\ln(1 + a_s(Q^2)\beta_0 \ln \nu)}{(1 + a_s(Q^2)\beta_0 \ln \nu)}$$
$$+O(a_s^2(a_s \ln \nu)^n). \tag{4.4.3}$$

Therefore equation (4.4.1) becomes

$$\ln \tilde{\sigma}_q^{[a]}(N, a_s(Q^2)) = \int_0^1 dz \frac{z^{N-1} - 1}{1 - z}\Bigg[\frac{a\tilde{P}^{(0)}}{\beta_0}\ln(1 + a_s(Q^2)\beta_0 \ln(1 - z)^a)$$
$$+\frac{a\tilde{P}^{(0)}\beta_1}{\beta_0^3}\beta_0 a_s(Q^2)\frac{\ln(1 + a_s(Q^2)\beta_0 \ln(1 - z)^a)}{(1 + a_s(Q^2)\beta_0 \ln(1 - z)^a)}$$
$$+\left(\frac{f^{[a](1),0}}{\beta_0} + \frac{a\tilde{P}^{(0)}\beta_1}{\beta_0^3} - \frac{a\tilde{P}^{(1)}}{\beta_0^2}\right)\frac{a_s(Q^2)\beta_0}{(1 + a_s(Q^2)\beta_0 \ln(1 - z)^a)}$$
$$+\left(\frac{a\tilde{P}^{(1)}}{\beta_0} - \frac{a\tilde{P}^{(0)}\beta_1}{\beta_0^2}\right)a_s(Q^2) + O(a_s^2(a_s \ln(1 - z)^a)^m)\Bigg]. \tag{4.4.4}$$

To do the Mellin transforms, we use the result that

$$\int_0^1 dz \frac{z^{N-1} - 1}{1 - z}\ln^k(1 - z) = (-1)^{k+1}\left(\frac{\ln^{k+1} N}{k + 1} + \gamma_E \ln^k N\right) + O(\ln^{k-1} N). \tag{4.4.5}$$

Therefore for a general function $f(A \ln(1 - z))$ (where A plays the role of $a_s\beta_0$),

$$\int_0^1 dz \frac{z^{N-1} - 1}{1 - z} f(A \ln(1 - z)^a) = -\left[\frac{\ln N}{A}\int_0^A dA + \gamma_E\right] f(-A \ln N^a)$$
$$+O(A(A \ln N^a)). \tag{4.4.6}$$

Equation (4.4.6) was obtained by writing f as an infinite series in its argument, using equation (4.4.5) and then resumming the series. Using equation (4.4.6), neglecting all terms of $O(A(A\ln N^a))$ and higher since we do not need them, we find

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} \ln(1+A\ln(1-z)^a) =$$

$$\frac{1}{aA}\left[(1-A\ln N^a)\ln(1-A\ln N^a)+A\ln N^a\right]-\gamma_E\ln(1-A\ln N^a) \tag{4.4.7}$$

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\frac{A}{1+A\ln(1-z)^a}\right] = \frac{1}{a}\ln(1-A\ln N^a) \tag{4.4.8}$$

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z}\left[\frac{A\ln(1+A\ln(1-z)^a)}{1+A\ln(1-z)^a}\right] = \frac{1}{2a}\ln^2(1-A\ln N^a) \tag{4.4.9}$$

$$\int_0^1 dz \frac{z^{N-1}-1}{1-z} A = -A\ln N. \tag{4.4.10}$$

Therefore equation (4.4.4) becomes

$$\ln\tilde{\sigma}_q^{[a]}(N,a_s(Q^2))$$

$$\begin{aligned}
&= \frac{\tilde{P}^{(0)}}{\beta_0}\frac{1}{a_s(Q^2)\beta_0}\left[(1-a_s(Q^2)\beta_0\ln N^a)\ln(1-a_s(Q^2)\beta_0\ln N^a)+a_s(Q^2)\beta_0\ln N^a\right] \\
&\quad+\left(\frac{f^{[a](1),0}}{a\beta_0}-\frac{a\tilde{P}^{(0)}\gamma_E}{\beta_0}+\frac{\tilde{P}^{(0)}\beta_1}{\beta_0^3}-\frac{\tilde{P}^{(1)}}{\beta_0^2}\right)\ln(1-a_s(Q^2)\beta_0\ln N^a) \\
&\quad+\frac{\tilde{P}^{(0)}\beta_1}{2\beta_0^3}\ln^2(1-a_s(Q^2)\beta_0\ln N^a)-\left(\frac{\tilde{P}^{(1)}}{\beta_0^2}-\frac{\tilde{P}^{(0)}\beta_1}{\beta_0^3}\right)a_s(Q^2)\beta_0\ln N^a \\
&\quad+O(a_s(a_s\ln N)^m).
\end{aligned} \tag{4.4.11}$$

We now determine $f^{[a](1),0}$, which requires expanding equation (4.4.11) in $a_s(Q^2)$ to NLO and comparing with the NLO result. However, we would like to compare with recent 2 loop results as a check on equation (4.2.34), so we will expand to NNLO. The result is

$$\ln \widetilde{\sigma}_q^{[a]}(N, a_s(Q^2)) = a_s(Q^2)\left(\frac{1}{2}a^2\widetilde{P}^{(0)}\ln^2 N + \left(a^2\widetilde{P}^{(0)}\gamma_E - f^{[a](1),0}\right)\ln N\right)$$
$$+a_s^2(Q^2)\left(\frac{1}{6}a^3\widetilde{P}^{(0)}\beta_0\ln^3 N + \left(\frac{1}{2}a^3\beta_0\widetilde{P}^{(0)}\gamma_E - \frac{1}{2}a\beta_0 f^{[a](1),0} + \frac{1}{2}a^2\widetilde{P}^{(1)}\right)\ln^2 N\right)$$
$$+O(a_s^3). \tag{4.4.12}$$

Now if

$$\widetilde{\sigma}_q^{[a]}(N, a_s(Q^2)) = 1 + \widetilde{\sigma}_q^{[a](1)}a_s(Q^2) + \widetilde{\sigma}_q^{[a](2)}a_s^2(Q^2) + O(a_s^3), \tag{4.4.13}$$

then

$$\ln \widetilde{\sigma}_q^{[a]}(N, a_s(Q^2)) = \widetilde{\sigma}_q^{[a](1)}a_s(Q^2) + \left(\widetilde{\sigma}_q^{[a](2)} - \frac{1}{2}\left(\widetilde{\sigma}_q^{[a](1)}\right)^2\right)a_s^2(Q^2) + O(a_s^3). \tag{4.4.14}$$

We first examine the case of DIS, where $a = 1$. For large N, from [57, 35]

$$\widetilde{\sigma}_q^{[1](1)}(N, a_s(Q^2)) = C_F\ln^2 N + C_F\left(2\gamma_E + \frac{3}{2}\right)\ln N + \gamma_E^2 + \frac{3}{2}\gamma_E - \zeta(2) + O\left(\frac{1}{N}\right) \tag{4.4.15}$$

$$\widetilde{\sigma}_q^{[1](2)}(N, a_s(Q^2)) = \left(2C_F^2\gamma_E - \frac{2}{9}C_F T_R n_f + \frac{3}{2}C_F^2 + \frac{11}{18}C_F C_A\right)\ln^3 N$$
$$+\left(\frac{11}{6}C_F C_A\gamma_E + \frac{9}{2}C_F^2\gamma_E - C_F C_A\zeta(2) - \frac{27}{8}C_F^2 - C_F^2\zeta(2) + \frac{367}{72}C_F C_A + 3C_F^2\gamma_E^2\right.$$
$$\left.-\frac{2}{3}C_F T_R n_f\gamma_E - \frac{29}{18}C_F T_R n_f\right)\ln^2 N + O(\ln N), \tag{4.4.16}$$

so that

$$\ln \widetilde{\sigma}_q^{[1]}(N, a_s(Q^2)) = \left(C_F\ln^2 N + \left(2C_F\gamma_E + \frac{3}{2}C_F\right)\ln N + O(1)\right)a_s(Q^2)$$

$$+\Bigg(\left(-\frac{2}{9}C_F T_R n_f + \frac{11}{18}C_F C_A\right)\ln^3 N + \left(\frac{11}{6}C_F C_A \gamma_E - C_F C_A \zeta 2 + \frac{367}{72}C_F C_A - \frac{2}{3}C_F T_R n_f \gamma_E - \frac{29}{18}C_F T_R n_f\right)\ln^2 N + O(\ln N)\Bigg) a_s^2(Q^2) + O(a_s^3). \tag{4.4.17}$$

Comparing equations (4.4.17) and (4.4.12), we find

$$\tilde{P}_{qq}^{V(0)} = 2C_F \tag{4.4.18}$$

$$\tilde{P}_{qq}^{V(1)} = C_F C_A\left(\frac{67}{9} - 2\zeta(2)\right) - \frac{20}{9}C_F T_R n_f \tag{4.4.19}$$

as we must have [59], and so

$$f^{[1](1),0} = -\frac{3}{2}C_F. \tag{4.4.20}$$

Now we consider Drell-Yan, where $a = 2$. For large N, from [46],

$$\tilde{\sigma}_q^{[2](1)}(N, a_s(Q^2)) = 4C_F \ln^2 N + 8C_F\gamma_E \ln N + 4C_F\gamma_E^2 + 8C_F\zeta(2) - 8C_F + O(1) \tag{4.4.21}$$

$$\tilde{\sigma}_q^{[2](2)}(N, a_s(Q^2)) = 8C_F^2 \ln^4 N + \left(\frac{44}{9}C_F C_A - \frac{16}{9}C_F T_R n_f + 32C_F^2\gamma_E\right)\ln^3 N$$
$$+\Bigg(-\frac{40}{9}C_F T_R n_f - \frac{16}{3}C_F T_R n_f \gamma_E + \frac{44}{3}C_F C_A \gamma_E + \frac{134}{9}C_F C_A - 4C_F C_A \zeta(2)$$
$$+32C_F^2\zeta(2) + 48C_F^2\gamma_E^2 - 32C_F^2\Bigg)\ln^2 N + O(\ln N), \tag{4.4.22}$$

so that

$$\ln \tilde{\sigma}_q^{[2]}(N, a_s(Q^2)) = \left(4C_F \ln^2 N + 8C_F\gamma_E \ln N + O(1)\right) a_s(Q^2)$$
$$+\Bigg(\left(\frac{44}{9}C_F C_A - \frac{16}{9}C_F T_R n_f\right)\ln^3 N + \Bigg(-\frac{40}{9}C_F T_R n_f - \frac{16}{3}C_F T_R n_f \gamma_E + \frac{44}{3}C_F C_A \gamma_E$$

$$+\frac{134}{9}C_F C_A - 4C_F C_A \zeta(2)\Big) \ln^2 N + O(\ln N)\Big) a_s^2(Q^2) + O(a_s^3). \qquad (4.4.23)$$

Comparing equations (4.4.23) and (4.4.12), we find equations (4.4.18) and (4.4.19) hold again, as they should, so that

$$f^{[2](1),0} = 0. \qquad (4.4.24)$$

Furthermore, with these results for the $f^{[a](1),0}$, equation (4.4.11) agrees with the results in [51].

Note that equation (4.4.15) and must be subtracted from the coefficient function when adding equation (4.4.11) to the coefficient function. Likewise equation (4.4.21) must be subtracted from the coefficient function when adding equation (4.4.11) to the coefficient function.

The ratio of the NLO+NLL resummed to NLO unresummed factorized non singlet quark cross sections in equation (4.4.11) for $Q^2 = 100$ GeV^2 are compared in figures 4.1 and 4.2. (F_2 is used in the DIS case). Note that the Landau poles are higher than the maximum value of N (10) in both graphs. In both cases the resummed and unresummed expressions are equal at $N = 1$, so that the Adler sum rule of section 2.21 is obeyed in the resummed case as it is in the unresummed case. These graphs show that resummation causes both cross sections to rise much faster than in the unresummed case. Resummation makes a much larger difference to the Drell-Yan cross section than to the DIS cross section, possibly because of more incoming partons contributing to the large N cross section.

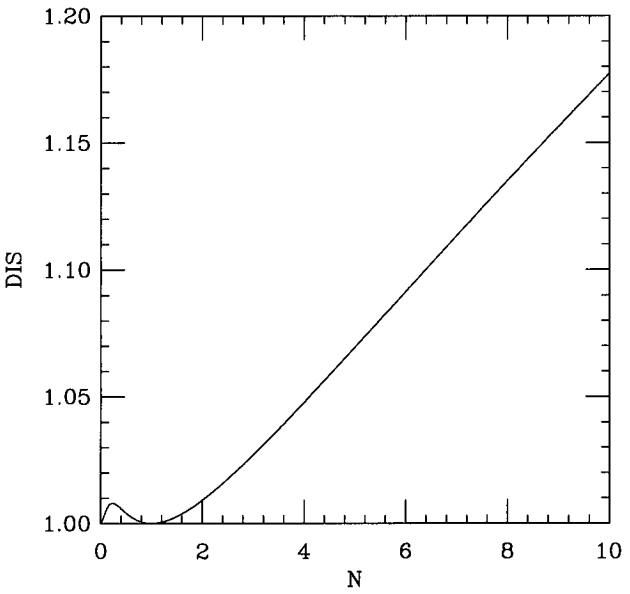

Figure 4.1: Quark initiated DIS cross section.

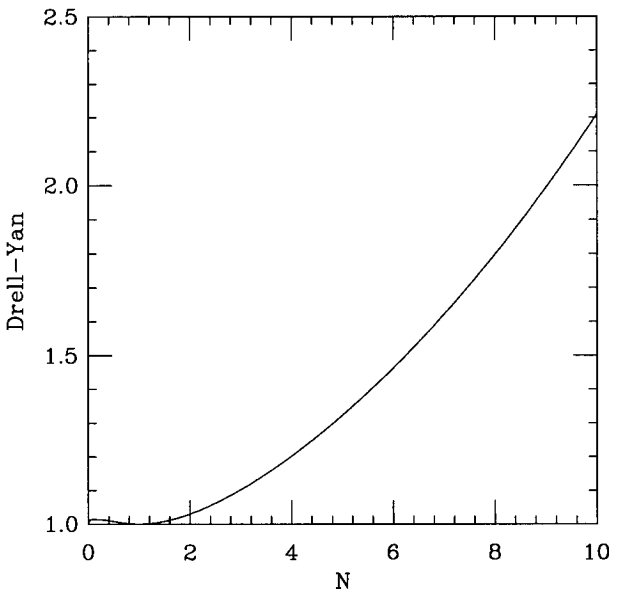

Figure 4.2: Quark initiated DY cross section.

4.5 Factorization And Renormalization Scale Dependence

Suppose we change μ_F from $\mu_F^2 = Q^2$ to $\mu_F^2 = k_F Q^2$. Then, from equation (4.2.8), the change to $\ln \tilde{\sigma}_q^{[a]}$ is

$$\ln \tilde{\sigma}_q^{[a]} \to \ln \tilde{\sigma}_q^{[a]} - a \int_0^1 dz z^{N-1} \int_1^{k_F} \frac{d\lambda}{\lambda} a_s(\lambda Q^2, \epsilon) P_{(+)}^{\sigma_q}(z, a_s(\lambda Q^2, \epsilon)). \tag{4.5.1}$$

We shall only be interested in resumming LL and NLL terms here. In this case we can write

$$\ln \tilde{\sigma}_q^{[a]} = \ln N^a \tilde{g}_{-1}^{[a]}(a_s(Q^2)\beta_0 \ln N^a) + \tilde{g}_0^{[a]}(a_s(Q^2)\beta_0 \ln N^a). \tag{4.5.2}$$

Then after the change given in equation (4.5.1), we obtain

$$\begin{aligned} \ln \tilde{\sigma}_q^{[a]} = \ln N^a \tilde{g}_{-1}^{[a]}(a_s(Q^2)\beta_0 \ln N^a) + \tilde{g}_0^{[a]}(a_s(Q^2)\beta_0 \ln N^a) \\ + \frac{\tilde{P}_{qq}^{V(0)}}{\beta_0} \ln k_F a_s(Q^2)\beta_0 \ln N^a, \end{aligned} \tag{4.5.3}$$

We now change the renormalization scale from $\mu_R^2 = Q^2$ to $\mu_R^2 = k_R Q^2$. In other words all occurrences of $a_s(Q^2)$ are replaced with

$$a_s(Q^2) \to a_s(k_R Q^2) + a_s^2(k_R Q^2)\beta_0 \ln k_R. \tag{4.5.4}$$

Firstly,

$$\begin{aligned} \tilde{g}_0^{[a]}(a_s(Q^2)\beta_0 \ln N^a) + \frac{\tilde{P}_{qq}^{V(0)}}{\beta_0} \ln k_F a_s(Q^2)\beta_0 \ln N^a \\ \longrightarrow \tilde{g}_0^{[a]}(a_s(k_R Q^2)\beta_0 \ln N^a) + \frac{\tilde{P}_{qq}^{V(0)}}{\beta_0} \ln k_F a_s(k_R Q^2)\beta_0 \ln N^a, \end{aligned} \tag{4.5.5}$$

since the difference between the right and left hand side is of $O(a_s(a_s \ln N)^n)$. However, the function $\tilde{g}_{-1}$ does change, in the manner

$$\ln N^a \tilde{g}_{-1}^{[a]}(a_s(Q^2)\beta_0 \ln N^a) \longrightarrow \ln N^a \tilde{g}_{-1}^{[a]}((a_s(k_R Q^2) + a_s^2(k_R Q^2)\beta_0 \ln k_R)\beta_0 \ln N^a)$$

$$= \ln N^a \tilde{g}_{-1}^{[a]}(a_s(k_R Q^2)\beta_0 \ln N^a) + \ln k_R \left(a_s(k_R Q^2)\beta_0 \ln N^a\right)^2 \frac{d\tilde{g}_{-1}^{[a]}(x)}{dx}\Big|_{x=a_s(k_R Q^2)\beta_0 \ln N^a}$$

$$= \ln N^a \tilde{g}_{-1}^{[a]}(a_s(k_R Q^2)\beta_0 \ln N^a)$$

$$- \frac{\tilde{P}_{qq}^{V(0)} \ln k_R}{\beta_0} \left(\ln(1 - a_s(k_R Q^2)\beta_0 \ln N^a) + a_s(k_R Q^2)\beta_0 \ln N^a\right). \qquad (4.5.6)$$

Thus, finally,

$$\ln \tilde{\sigma}_q^{[a]}(N, a_s(Q^2))$$

$$= \frac{\tilde{P}^{(0)}}{\beta_0} \frac{1}{a_s(k_R Q^2)\beta_0} \Big[(1 - a_s(k_R Q^2)\beta_0 \ln N^a) \ln(1 - a_s(k_R Q^2)\beta_0 \ln N^a)$$

$$+ a_s(k_R Q^2)\beta_0 \ln N^a\Big]$$

$$+ \left(\frac{f^{[a](1),0}}{a\beta_0} - \frac{a\tilde{P}^{(0)}\gamma_E}{\beta_0} + \frac{\tilde{P}^{(0)}\beta_1}{\beta_0^3} - \frac{\tilde{P}^{(1)}}{\beta_0^2}\right) \ln(1 - a_s(k_R Q^2)\beta_0 \ln N^a)$$

$$+ \frac{\tilde{P}^{(0)}\beta_1}{2\beta_0^3} \ln^2(1 - a_s(k_R Q^2)\beta_0 \ln N^a) - \left(\frac{\tilde{P}^{(1)}}{\beta_0^2} - \frac{\tilde{P}^{(0)}\beta_1}{\beta_0^3}\right) a_s(k_R Q^2)\beta_0 \ln N^a$$

$$- \frac{\tilde{P}_{qq}^{V(0)} \ln k_R}{\beta_0} \left(\ln(1 - a_s(k_R Q^2)\beta_0 \ln N^a) + a_s(k_R Q^2)\beta_0 \ln N^a\right).$$

$$+ \frac{\tilde{P}_{qq}^{V(0)}}{\beta_0} \ln k_F a_s(k_R Q^2)\beta_0 \ln N^a + O(a_s(a_s \ln N)^m). \qquad (4.5.7)$$

4.6 Results

In this section we fit PDF's to data, and use these results to predict upgraded TEVATRON (proton-antiproton scattering) and future LHC (proton-proton scattering) data. The upgraded TEVATRON will operate at a centre of momentum energy $\sqrt{s} = 2$ TeV, and the future LHC at $\sqrt{s} = 14$ TeV. Our aim will be to

determine what differences and improvements soft resummation makes to predictions and fits.

At large ξ the quark PDF's go to zero so that it would seem that soft resummation would not make any difference. It has been argued in [60] that resummation is unlikely to make a serious difference to global fits of PDF's to DIS data as long as the region with $Q^2 \lesssim 10$ GeV^2 and very large η is avoided. However, the errors at large η in fitted PDF's, due to truncation of perturbation series, propagate to lower values of η when PDF's are evolved up to Q^2. Thus it is important to investigate whether soft gluon resummation could make a difference at intermediate values of η, and to see over what range of η in DIS and τ in Drell-Yan the theoretical error is reduced when using resummation.

We fit singlet and non singlet PDF's to proton and deuteron BCDMS F_2 data [61, 62], proton and deuteron SLAC F_2 data [63] and proton H1 F_2 data [64]. BCDMS and SLAC is typically in the range $0.1 \lesssim x \lesssim 1$ and H1 data $10^{-5} \lesssim x \lesssim 0.1$. The BCDMS and SLAC data sets contain some of the largest η data available. The reason for using H1 data is because of the following. The resummation should not affect the PDF's at small ξ, but in practice it does: Soft resummation results in a lower singlet at large ξ, and so as a result of the momentum sum rule the gluon and singlet PDF's might be larger at small ξ. Note that fitting only to large η data will constrain the PDF's less at small ξ than at large ξ. Thus to make the PDF's at small ξ fitted with resummation similar to those fitted without resummation, we also include proton H1 F_2 data in the fit. In principle this does not matter as far as obtaining PDF's goes, since the variation in the PDF's at small η that would be obtained in a fit that excluded H1 data for different values of the renormalization and factorization scales would give us some idea of the error on the PDF's at small ξ that are caused by possible large theoretical errors at small η. However, our reason for including H1 data is to determine whether soft resummation at large η interferes with the small η region - we are not concerned with small η uncertainties.

We use the statistical correlations between the data points within the BCDMS and SLAC data sets, which have been studied in [65, 66]. These were treated using the method described in appendix A. This appendix draws on some results

from [67, 68]. Since H1 data was used only to fix the low ξ behaviour of the PDF's, we simply combined the systematic and statistical errors in quadrature for the H1 data set.

We fit valence quark PDF's to CCFR iron F_3 data [69, 70, 71], using the covariance matrix.

$\alpha_s(M_Z^2)$ is fixed to the world average [72],

$$\alpha_s(M_Z^2) = 0.118.$$

Note that a precise value for $\alpha_s(M_Z^2)$ is not quite as important as precision in the PDF's for making LHC and TEVATRON predictions [73]. (All other parameters, e.g. electroweak parameters, are sufficiently well known). However, note therefore that correlations between $\alpha_s(M_Z^2)$ and the PDF's will have to be neglected.

Since resummation is important at both large N and small Q^2, it is expected to make a substantial difference to the large η region of the fitted higher twist, which from section 2.5 is of $O(\Lambda_{QCD}^2/Q^2)$ and therefore contributes significantly to the overall DIS cross section at small Q^2. When fitting higher twist, it is also necessary to account for hadron and quark masses, since these give contributions of $O(M^2/Q^2)$, where M is the relevant mass. Thus we include hadron mass effects, using the method outlined in section 2.20. However, we neglect quark mass effects since these do not contribute at large η.

To take into account the theoretical uncertainties due to truncation, we vary the factorization and renormalization scales using the results of section 2.18.

The singlet and non singlet PDF's that were fitted to F_2 data were

$$T_3 = u^+ - d^+ \qquad T_8 = u^+ + d^+ - 2s^+ \tag{4.6.8}$$

$$\Sigma = u^+ + d^+ + s^+ \qquad G = g. \tag{4.6.9}$$

The parameterization used for these PDF's was

$$f(x) \propto x^a(1-x)^b(1+cx+d\sqrt{x}), \tag{4.6.10}$$

with the normalization

$$A = f(N=2).$$

We use the momentum sum rule to set $A_\Sigma = 1 - A_G$.

The valence quark PDF's that were fitted to F_3 data were

$$V_I^- = f_I^-, \; I = u, d$$

The parameterization used for these PDF's was

$$f(x) \propto x^a(1-x)^b, \tag{4.6.11}$$

since this parameterization was adequate for this data. We use the valence sum rules for the proton:

$$\int_0^1 dx(u(x) - \bar{u}(x)) = 2 \qquad \int_0^1 dx(d(x) - \bar{d}(x)) = 1 \tag{4.6.12}$$

$$\int_0^1 dx(s(x) - \bar{s}(x)) = 0. \tag{4.6.13}$$

Furthermore, the condition

$$s(x, \mu_F^2) = \bar{s}(x, \mu_F^2)$$

is assumed to hold. The same condition is made to hold for heavy quarks. Heavy quarks are set to zero below threshold, and evolve from zero at threshold. The thresholds are taken as $m_c' = 1.5$ GeV and $m_b' = 5$ GeV. Isospin invariance is

assumed to hold between the neutron and proton. We set $Q_0^2 = 4$ GeV^2. All data was cut at

$$Q^2 > 4 \text{ GeV}^2 \qquad\qquad W^2 > 4 \text{ GeV}^2.$$

Higher twist effects were incorporated by taking the fitted $F_{2,3}$ as

$$F_{2,3}(\eta, Q^2) = \left(1 + \frac{H_{2,3}(\eta)}{Q^2}\right) F_{2,3}^{L.T.}(\eta, Q^2),$$

where

$$H_2(\eta) = A_{H_2}\eta^{a_{H_2}}(1-\eta)^{b_{H_2}}(1 + c_{H_2}\eta + d_{H_2}\sqrt{\eta}) \tag{4.6.14}$$

$$H_3(\eta) = A_{H_3}\eta^{a_{H_3}}(1-\eta)^{b_{H_3}} \tag{4.6.15}$$

and $F_{1,2,3}^{L.T.}(\eta, Q^2)$ is the leading twist part of $F_{1,2,3}(\eta, Q^2)$. The simpler parameterization of equation (4.6.15) for $H_3(\eta)$ was used since the parameterization of the form in equation (4.6.14) for $H_3(\eta)$ did not give any improvement in the fits to the F_3 data.

The evolution part of the DIS code was checked using [74].

χ^2 was minimized using the CERNLIB MINUIT package. The results for the parameters used in the fit to F_3 data are shown in tables D.1.2 to D.1.20. The results for the parameters used in the fit to F_2 data are given in tables D.1.1 to D.1.19. Only the diagonal components of the covariance matrix have been presented.

Figures D.2.1 to D.2.10 show the higher twist fits for F_2, and figures D.3.1 to D.3.10 show the higher twist fits for F_3.

The code we use to generate Drell-Yan differential cross sections was tested with the MRST99 parameters in [26], and the results for the total cross sections for lepton decay agreed well with [73]. These results, of our code with the MRST99

parameters, are shown in figures D.4.1 to D.4.5 (where the cross sections have been multiplied by their branching ratios, so these cross sections are for dilepton production). Note that, unlike [26], we choose to consider W^+ and W^- production separately, purely for the benefit of having more cross sections to consider. The total cross sections with our parameters are shown in figures D.5.1 to D.5.5, using both sets of PDF's and using both the resummed and unresummed DY cross sections. The notation that is used to label these 4 scenarios is explained in table 4.1. The errors in the DY cross sections that result from the (correlated) errors in the fitted PDF's have also been shown at some values of $\sqrt{s}$. For clarity, error bars have been offset, and each cluster of four error bars refers to a single x-axis position, this being the average of the four x-axis positions. They appear in the order 00, 01, 10, 11. Note that the cross section for TEVATRON W^+ production is identical to that for TEVATRON W^- production, by charge conjugation symmetry.

DIS resummation	DY resummation	Label (XY)
Yes	Yes	11
No	Yes	01
Yes	No	10
No	No	00

Table 4.1: Labelling for the use of resummation in LHC predictions.

We examine the resulting predictions for DY differential cross sections as defined in equations (3.1.13) and (3.1.14). The predictions for these cross sections are shown in figures D.6.1 to D.6.5.

We also examine the ratios of the 01, 10 and 11 differential cross sections to the 00 differential cross section, shown in figures D.7.1 to D.7.5. The "XY" labels on this graphs are in the order the corresponding curves occur along the y-axis at the x-axis position of the labels.

Then, we set

$$\mu_F^2 = k_F Q^2 \qquad \mu_R^2 = k_R Q^2 \tag{4.6.16}$$

and take the values 4 and $\frac{1}{4}$ for both k_F and k_R. Each resulting PDF fit is used to calculate the Drell-Yan differential cross section for W^+, W^- and Z production at the LHC and TEVATRON, with the scales set to

$$\mu_F^2 = k_F Q^2 \qquad \mu_R^2 = k_R Q^2, \tag{4.6.17}$$

and k_F and k_R are taken to be the same values as those taken for the PDF's used in the given Drell-Yan calculation when fitting these to DIS data. This must be the case for k_F, so that the scale that the PDF's are defined at, μ_{F0}, is the same as the starting scale in the Drell-Yan calculation, but note that this does not have to be the case for k_R. We make this choice simply because it is the most obvious one. Note of course that Q^2 in Drell-Yan is not related to Q^2 in DIS. The notation for these 4 scenarios is shown in table 4.2.

k_R	k_F	Label (XY)
$\frac{1}{4}$	$\frac{1}{4}$	55
$\frac{1}{4}$	4	52
4	$\frac{1}{4}$	25
4	4	22

Table 4.2: Values of XY for σ_{XY} and the corresponding scale variations.

The results of the scale variations for the 00 and 11 cases are shown in figures D.8.1 to D.8.10, and are presented as ratios over the cross section calculated with $k_F = k_R = 1$, which is 00 or 11 in correspondence with the other four. For each cross section, the y range for the 11 case is the same as that for the 00 case. The "XY" labels on this graphs are in the order the corresponding curves occur along the y-axis at the x-axis position of the labels.

The relatives theoretical errors between the 00 and 11 cases are shown in figure 4.3. These were calculated by taking the highest and lowest cross sections of all scale variations for each of the three specified values of τ.

Process	% reduction at $\tau =$		
	10^{-1}	10^{-2}	10^{-4}
LHC Z	18	25	80
LHC W^+	23	48	54
LHC W^-	-	-	55
TEVATRON Z	17	17	50
TEVATRON $W^{+/-}$	52	49	84

Table 4.3: Percentage reduction in the theoretical error of the resummed cross sections compared to the unresummed ones.

4.7 Conclusions

For the cases $k_R = k_F$, the F_2 higher twist plots are reduced at large η when resummation is used compared to when conventional NLO calculations are used. However, for the 52 case, the higher twist fit is slightly larger at large η in the case that resummation is used compared to the case with no resummation, and for the 25 case the higher twist fit is much larger at large η when resummation is used. However, we expect the cases $k_R \neq k_F$ to be the least physical cases. Indeed, the 25 case, the most serious deviation from the rest of the higher twist fits, gives a negative higher twist whereas all the other fits indicate a positive higher twist is more likely. Thus without resummation, higher twist diverges at large η to reproduce the resummed cross section at large η. It may be that a suitably divergent higher twist combined with the unresummed NLO cross section reproduces the resummed cross section better than the LL+NLL resummed cross section with small higher twist at large η, since the fits to F_2 data generally give a lower χ^2 when resummation is not used. It would be interesting to extend the analysis to NNLO+NNL.

In the case of F_3, resummation generally improves the fit. Not much can be said about the higher twist, since the errors are much larger than the central values, and the higher twist is perhaps too constrained by the relatively simple parameterization used. It would be interesting to consider the effect of using larger and / or more accurate data sets that constrain valence quark PDF's.

Our disagreements with the MRST99 Drell-Yan predictions may be due to the MRST99 singlet and non singlet PDF's being parameterized differently and with different constraints; The PDF's are parameterized as sea $(2\bar{u} + 2\bar{d} + 2\bar{s})$ and $\bar{d} - \bar{u}$, and furthermore some constraints are put on s ($s = \frac{1}{4}(\bar{u} + \bar{d})$). It would be interesting to examine the dependence of Drell-Yan cross sections on PDF parameterization by trying as many different types of parameterization for the PDF's as possible. They fit to a larger amount of data, and systematic and statistical errors are added in quadrature (except for F_3). The MRST99 set also uses $Q_0^2 = 1$ GeV. In common with us, the gluon and valence quark PDF's are parameterized the same way and the momentum sum rule, flavour sum rules and the condition $s = \bar{s}$ are imposed.

The errors on the Drell-Yan total cross sections due to the errors on the PDF's are typically around $\pm 2\%$, in contrast to [73] where the errors are around $\pm 5\%$ (although they also included other sources of error, such as variations with respect to $\alpha_s(M_Z^2)$ and errors on electroweak parameters). However, in our case the error matrix for the PDF parameters was not always positive definite, so that our errors should only be interpreted as an order of magnitude. A non positive definite error matrix indicates that the fitted parameters are not sufficiently close to the minimum in χ^2 for a calculation of this matrix to give sensible answers, or more drastically that a saddle point has been reached. We suspect the former is the case, since the square of the errors on the Drell-Yan cross sections were always positive, indicating that the the non positive definiteness was only slight. Furthermore, the exact position of the minimum is not important as long as the fitted parameters are close to the minimum, and most importantly that $\tilde{\chi}^2 \lesssim 1$.

The errors from the PDF's on the Drell-Yan differential cross sections increased from $\sim 1\%$ at $\tau \sim 10^{-6}$ to $\sim 10\%$ at $\tau \sim 10^{-1}$.

The LHC W^- differential cross section is qualitatively different from the other cross sections, the biggest difference being that we get a negative cross section for $\tau \gtrsim 0.1$. This is not against physical expectations since the overall uncertainty in the cross section may be larger than the cross section itself, but the negativity of this particular cross section in this region makes it difficult to analyse the effect of resummation here.

At small τ, the 00, 01, 10 and 11 converge quite well, a result of using H1 data in the PDF fitting, which constrains the PDF's to be similar at small ξ. It is a little surprising that the 10 and 11 curves are lower than the 00 and 01 curves, since resummation in DIS pushes the singlet down at large ξ, and thus at small ξ one would expect that because of the momentum sum rule that resummation in DIS pushes the singlet and gluon up, and therefore pushes the Drell-Yan cross section up at small τ, but this is not the case.

Except for the LHC W^- differential cross section, at large τ, resummation in Drell-Yan alone causes a rise in the Drell-Yan cross section over the 00 case, whereas using resummed PDF's (but no resummation in Drell-Yan) causes the Drell-Yan cross section to fall, since the PDF's at large ξ are suppressed due to the larger DIS coefficient functions in the DIS case. The net effect, i.e. the 11 case, is a cancellation effect in which the only real deviation from the 00 case is for $\tau \gtrsim 0.1$, where the cross section is generally lower by about a factor of ~ 0.5. Note the biggest effect of resummation on Drell-Yan cross sections is in the PDF's, which is unexpected since resummation tends to affect Drell-Yan cross sections much more than DIS ones. It would be interesting to include Drell-Yan data in the PDF fitting.

Scale variations show that theoretical errors become larger as τ increases. The cross sections are most out of control when $k_R \neq k_F$. Resummation does help to reduce this, but the errors are still large at large τ. It would be interesting to examine the theoretical errors (scale dependence) on the Drell-Yan cross sections from the PDF's alone, and from the Drell-Yan cross sections alone. This would require varying μ_F, while keeping μ_{F0} fixed.

Chapter 5

An Improved Treatment Of Heavy Quarks

The ZM-VFNS mentioned in section 2.19 does not deal accurately with the case $Q \sim m$, where m is any parton mass. We will now examine what happens when quark masses *and* all known quarks are included in QCD applied to high energy processes. We will focus on DIS, but the ideas are applicable to other processes.

5.1 Bare Quantities

Our aim in this section is to first work out how to write hadronic cross sections in terms of bare quantities which can be expressed in terms of Feynman diagrams. Then in later sections we shall consider how to reorganise the cross sections such that they can be expressed in a phenomenological useful way, i.e. such that they are expressed only in terms of quantities that we can approximate well using perturbation theory.

The mass of a particle depends on the scheme used to define the mass. The *pole mass* of a particle is the mass of a particle external to a Green's function that generates a pole in this Green's function, and is also its physical mass. We will choose to express all masses as pole masses, and the pole mass of a parton i will

be denoted m_i. All partons, including the gluon, will be given a mass. A lower case index such as i can take on the values $i = 0, ..., 6$, and an uppercase index I can only take on the values $I = 1, ..., 6$, i.e. it refers only to quarks. Partons are ordered according to their masses (except for u and d, which are interchanged), e.g. $i = 0$ refers to the gluon and $i = 6$ refers to the top quark.

The discussion in section 2.5 can be reapplied in the case that the partons each have a mass, and we include all partons with mass greater than of $O(Q)$. However, it is convenient to modify the $\hat{O}$ operator such that massive incoming partons are on shell, so that QCD symmetries remain intact, e.g. gauge invariance, and such that massless incoming partons are given a small mass to regularize mass singularities. In the collinear frame, since $\mathbf{p_T}$ can be set to zero for leading twist calculations, we can choose $\hat{O}$ to set

$$p = \left(\xi P^+, \frac{m^2}{2\xi P^+}, \mathbf{0}\right). \tag{5.1.1}$$

Note that this is the most general momentum for a particle of mass m moving along the 3-axis. Note from equation (5.1.1) that the modification of $\hat{O}$ only changes p^-, which does not contribute to the leading region as defined in equation (2.5.17), and so the arguments of section 2.5 are unchanged. Furthermore, we choose $\hat{O}$ to project out the physical polarizations for a parton with mass m. Again this only changes the spin components of diagrams that do not contribute to the leading region.

We also include quarks with mass greater than of $O(Q)$ in the theory, and we treat them the same way as we treated partons of mass less than of $O(Q)$ in section 2.5. In particular, our 2PI diagrams are now 2PI in *all* lines, including those with mass greater than of $O(Q)$, and $\hat{O}$ is the same (except for the mass m in its explicit definition) on *all* lines. Then r_Θ as defined by equation (2.5.18) is again of $O\left(\frac{\Lambda^2_{QCD}}{Q^2}\right)$ for the same reasons. We will take the number of flavours to be 6. Thus equation (2.1.3) becomes, in the case of photon scattering,

$$W_{\mu\nu}(P,q) = \sum_{i=0}^{6} \int_\eta^1 \frac{d\xi}{\xi} W^i_{\mu\nu}(p,q) f^+_{0i}(\xi). \tag{5.1.2}$$

The limits of the ξ integration in equation (5.1.2) will now be derived, using [45]. From equations (2.20.1) and (5.1.1), since the + component of what is left of the hadron after the initial parton has left it, $P^+(1-\xi)$, must be positive, this gives the upper limit on the ξ integration as $\xi \leq 1$. To find the lower limit on ξ, we consider a frame where the virtual photon - initial parton system has zero spatial momentum. Now, for a given diagram, the energy of the final state system in this frame obeys

$$p_0 + q_0 = \sum_{i \text{ final}} E_i. \tag{5.1.3}$$

But since

$$E_i = \sqrt{\mathbf{p}_i^2 + m_i^2} \qquad \sum_{i \text{ final}} \mathbf{p}_i = 0, \tag{5.1.4}$$

then

$$(p_0 + q_0)^2 \geq \left(\sum_{j \text{ final}} m_j \right)^2 = \hat{s}_{\text{th}}. \tag{5.1.5}$$

This last equation, together with equations (2.20.2) and (5.1.1), gives the lower limit on the ξ integration when convoluting the PDF of parton i with $W^i_{\mu\nu}$, for a given diagram labelled α:

$$\xi \geq \eta \chi_{i,\alpha}, \tag{5.1.6}$$

where

$$\chi_{i,\alpha} = \frac{1}{2} \left(1 + \frac{\hat{s}_{\text{th}i,\alpha} - m_i^2}{Q^2} + \sqrt{\left(1 + \frac{\hat{s}_{\text{th}i,\alpha} - m_i^2}{Q^2} \right)^2 + 4\frac{m_i^2}{Q^2}} \right). \tag{5.1.7}$$

Clearly $\chi_{i,\alpha}$ is different for different partons i and subprocesses α. Thus, rather than choose $\chi_{i,\alpha}$ as the lower limit for ξ, instead we set the range of integration

in ξ to always be

$$\eta \leq \xi \leq 1, \tag{5.1.8}$$

and for the different subprocesses α we absorb a factor $\theta\left(1-\chi_{i,\alpha}\frac{\eta}{\xi}\right)$ into $W^i_{\mu\nu}$. This range of integration is always larger than the size of the phase spaces for the individual subprocesses α, since $\chi_{i,\alpha} \geq 1$, and $\chi_{i,\alpha} = 1$ only when all parton masses are zero. Thus the integration limits in equation (5.1.2) follow.

We will only be interested in F_2 and virtual photon exchange in this chapter. We now find how to express F_2 in terms of bare PDF's. We will from now on use the shorthand

$$\eta' = \frac{\eta}{\xi}. \tag{5.1.9}$$

Using equations (1.2.18), (2.20.11) and (5.1.1), we find

$$\frac{1}{\rho^4}\left[\frac{12x^3}{Q^2}P^\mu P^\nu - x\rho^2 g^{\mu\nu}\right] W^i_{\mu\nu} = \xi\Omega(\eta,Q^2)X_i\left(\eta',Q^2\right)\widehat{F}_{2i}\left(\eta',Q^2\right) \tag{5.1.10}$$

where

$$X_i = \frac{\left(1+\eta'^2\frac{m_i^2}{Q^2}\right)^2}{\left(1-\eta'^2\frac{m_i^2}{Q^2}\right)}. \tag{5.1.11}$$

Note $\widehat{F}_{2i}$ depends on the m_i^2. It will be understood that all functions in this chapter will contain mass dependence unless otherwise stated, but this dependence will not be shown in the arguments for brevity. Thus from equations (5.1.10) and (5.1.2) we find

$$F_2^{Norm}(\eta,Q^2) = \sum_{i=0}^{6}\int_\eta^1 d\xi \widehat{S}_{2i}\left(\eta',Q^2\right) f_{0i}^+(\xi), \tag{5.1.12}$$

where

$$F_2^{Norm}(\eta, Q^2) = \frac{F_2(\eta, Q^2)}{\Omega(\eta, Q^2)} \tag{5.1.13}$$

$$\widehat{S}_{2i} = X_i \widehat{F}_{2i}. \tag{5.1.14}$$

In Mellin space, i.e. where

$$F_2^{Norm}(N-1, Q^2) = \int_0^1 d\eta \frac{\eta^{N-1}}{\eta} F_2^{Norm}(\eta, Q^2), \tag{5.1.15}$$

equation (5.1.12) becomes, omitting parton indices and the summation sign $\sum$ for brevity,

$$F_2^{Norm}(N-1, Q^2) = \widehat{S}_2(N-1, Q^2) f_0^+(N). \tag{5.1.16}$$

5.2 The Bare Perturbation Series With Heavy Quarks

The aim of factorization in chapter 2 was to separate hard from soft processes. In other words, the aim was to factor out $\ln^n \frac{Q^2}{\kappa^2}$ terms from the perturbative expansion of $W^i_{\mu\nu}$. In doing this, we assumed that Q was much greater than the masses of the first n_f quarks, so these masses could be neglected, and we assumed Q was much less than the remaining heavier quarks, so these heavier quarks could be neglected altogether. So all our expressions were chosen to be the ones in QCD for n_f massless quarks and a gluon only.

Suppose the first n_f quarks' masses were included in our expressions (but with Q still greater than of the order of all these masses), and the gluon was also given a small mass, but we still did not include quarks of mass greater than of $O(Q)$ in the theory. If we use the $\overline{MS}$ scheme again, a_s would still evolve according

to equation (2.4.2), with n_f flavours, where the β_i would be the same as in the massless theory. This mass independence is a consequence of the $\overline{MS}$ scheme - the $\overline{MS}$ counterterms are determined purely from the ultraviolet singular terms in cross sections, which must always be the same regardless of the masses involved. We will set $\mu_R = O(Q)$ to prevent terms of the form $\ln^j \frac{\mu_R^2}{Q^2}$ in the perturbation series from becoming large.

In giving each of these partons a mass, the $O\left(\ln^n \frac{Q^2}{\kappa^2}\right)$ terms disappear and we can set $\kappa = 0$. However, the perturbation series now contains terms that behave like $O\left(\ln^n \frac{Q^2}{m_i^2}\right)$, where $i = 0, ..., n_f$, as $m_i^2 \to 0$, which again arise from integration over the loops joining the 2PI diagrams discussed in section 2.5. (However the 2PI diagrams themselves are free of these large logarithms in an axial gauge.) While these terms are non-singular when parton i has a non-zero mass, they can still render the truncated series for $W^i_{\mu\nu}$ useless, and so still need to be factored out. Now, our only criterion for phenomenological factorized partonic cross sections will be that they are non singular as $m_i \to 0$ for $i = 0, ..., n_f$. Therefore we can regard parton masses as an alternative regulator of mass singularities, and so the factorization theorem of section 2.8 shows that all terms that are singular as $m_i \to 0$ for $i = 0, ..., n_f$ can be factored out of bare partonic cross sections. We will set $\mu_F = O(Q)$ to prevent terms of the form $\ln^j \frac{\mu_F^2}{Q^2}$ in the perturbation series from becoming large.

If we then also include the quarks with masses greater than of $O(Q)$, which we now wish to do, this would create more subtle problems. New UV divergences from diagrams with these quarks would appear which need to be subtracted. Suppose we renormalize in the $\overline{MS}$ scheme, so that a_s would evolve according to the $\overline{MS}$ scheme with 6 flavours. Then we would encounter *new* types of mass logarithms of the form $\ln^j \frac{\mu_R^2}{m_i^2}$, which, although suppressed by factors of $O\left(\left(\frac{m_i^2}{\mu_R^2}\right)^k\right)$ for $i = 0, ..., n_f$, are unsuppressed as $m_i^2 \to \infty$ for $i = n_f + 1, ..., 6$ and hence need to be resummed. It is important to note that these types of large mass logarithms are distinct from the potential mass singularities, of the form $\ln^j \frac{Q^2}{m_i^2}$, which are suppressed by factors of $O\left(\left(\frac{Q^2}{m_i^2}\right)^k\right)$ for $i = n_f + 1, ..., 6$ but not suppressed for $i = 0, ..., n_f$. To distinguish these two types of large mass logarithms, we shall

refer to the usual potential mass singularities, i.e. those terms of $O\left(\ln^j \frac{Q^2}{m_i^2}\right)$ for $i = 0, ..., n_f$, as *active logarithms* (AL's), and the large mass logarithms of $O\left(\ln^j \frac{m_I^2}{\mu_R^2}\right)$ for $I = n_f + 1, ..., 6$ as *non partonic logarithms* (NPL's). Since the inclusion of quarks of mass greater than of $O(Q)$ also introduces terms in cross sections that are non singular as these masses go to infinity, we will include terms of this type in our definition of NPL's.

The above paragraph goes against the physically motivated decoupling theorem discussed in chapter 2: As a quark mass goes to infinity, the quark should decouple from the theory, so there should be no large mass logarithms or finite contributions coming from a quark with mass greater than of $O(Q)$, and each term in the perturbation series should become that for just n_f flavours, not for 6 flavours. Clearly we need a new renormalization scheme in which, in the resulting perturbation series, these expectations are exhibited explicitly order by order.

When we then come to factorize, if Γ treats all 6 flavours in the same way then it would contain terms of $O\left(\ln^n \frac{\mu_F^2}{m_i^2}\right)$ for $i = 0, ..., n_f$, as is necessary to remove large mass logarithms from bare partonic cross sections, but it would also contain terms of $O\left(\ln^n \frac{\mu_F^2}{m_i^2}\right)$ for $I = n_f + 1, ..., 6$ which would have nothing to cancel in bare partonic cross sections, and so these latter terms in Γ would introduce unwanted large mass logarithms into the resulting factorized partonic cross sections' perturbation series. Thus we would need an alternative factorization scheme to the usual $\overline{MS}$ one.

5.3 Requirements For The New Scheme

When $O(m_{n_f}) < Q < O(m_{n_f+1})$, we would like a renormalization scheme where a_s evolves in exactly the same way as the $\overline{MS}$ a_s in a theory with only n_f flavours, and we would like a factorization scheme where the PDF's of partons with mass less than of $O(Q)$ evolve in exactly the same way as for $\overline{MS}$ factorized PDF's in a theory with n_f massless quarks. As we will see, this is in fact possible by exploiting the freedom in the choice of scheme for Z_{a_s} (the renormalization constant for a_{0s}) and Γ. Furthermore, we will see that in doing this the Z_{a_s} absorbs all NPL's and

Γ absorbs all AL's, which we can at present anticipate in the light of chapter 2. Then when a quark mass goes to infinity its decoupling from the theory is explicit at each order in the perturbation series, and when a quark mass goes to zero all terms in the perturbation series for a factorized partonic cross section remain non singular. The difference between the expressions for factorized partonic cross sections which include all quarks and their masses, in which renormalization has been carried out, and those corresponding ones in chapter 2 for n_f massless quarks only, order by order, should therefore consist only of terms of $O\left(\frac{m_i^2}{Q^2}\right)$ (or less) for $i = 0, ..., n_f$, and terms of $O\left(\frac{Q^2}{m_I^2}\right)$ (or less) for $I = n_f + 1, ..., 6$.

From [75] and [9], it is possible to perform a renormalization and factorization of this specification to all orders, using the n_fth *CWZ subscheme* for both, to be discussed in sections 5.4 and 5.5 respectively, where for $O(m_{n_f}) < Q < O(m_{n_f+1})$ all large mass logarithms, i.e. both AL's and NPL's, are absent at every order in the factorized partonic cross sections after renormalization has been carried out. The first n_f quarks and the gluon are essentially on the same footing as they were in chapter 2, and are called *active partons* (AP's). A quantity Q_i for $i = 0, ..., n_f$ will sometimes be written for brevity as Q_{AP}. Partons that are not AP's are called *non partonic quarks* (NPQ's). A quantity Q_I for $I = n_f + 1, ..., 6$ will sometimes be written for brevity as Q_{NPQ}. As all AP masses go to zero and all NPQ masses go to infinity, each term in the perturbation series for a hard partonic cross section, after the n_fth CWZ subschemes for renormalization and factorization have been applied, explicitly becomes what it would be in a theory with n_f massless quarks where the $\overline{MS}$ scheme has been used in both factorization and renormalization.

When Q goes from $O(m_{n_f}) < Q < O(m_{n_f+1})$ to $O(m_{n_f+1}) < Q < O(m_{n_f+2})$, we must change to the $n_f + 1$th CWZ subscheme for both renormalization and factorization to ensure that our truncated series is close to the all orders result, since otherwise the new NPL's and AL's that arise are not removed, and become larger as we go to higher orders in the perturbation series. One is free to choose the exact value of Q that determines when one should change subscheme, since when $Q = O(m_{n_f+1})$ the series for the factorized partonic cross section is free of large mass logarithms in both the n_fth and $n_f + 1$th CWZ subschemes for

renormalization and factorization. The lowest mass NPQ in the n_fth CWZ subscheme becomes the most massive AP in the $n_f + 1$th CWZ subscheme. If we have properly defined both schemes in terms of bare quantities, this will allow us to perform matching of quantities between schemes. This collection of CWZ subschemes, one for each value of n_f, is called the *variable flavour number scheme* (VFNS).

The VFNS was first formulated to NLO in [76], and proved to all orders in [9]. Explicit calculations to NLO for coefficient functions and splitting functions using the CWZ subschemes in both renormalization and factorization are given in [77]. Our intention here is to understand the technical procedure of removing all large mass logarithms to all orders when all quarks and their masses are included in the theory.

Since the gluon is massless and the u, d and s quarks all have mass less than or of the order of Λ_{QCD}, and since $Q \gg \Lambda_{QCD}$ in perturbative QCD, these particles should always be AP's. Furthermore, we can and shall set light parton masses to zero, because the contribution of light hadron masses to the hadronic cross section is of $O\left(\frac{m_i^2}{Q^2}\right)$ and we are neglecting terms of $O\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$. However, since the AL's are singular for light partons when the light parton masses are zero, we will only set light parton masses to zero once renormalization and factorization have been carried out. The heavy quark masses must be kept if we want our errors to be no more than of $O\left(\frac{\Lambda_{QCD}^2}{Q^2}\right)$.

5.4 CWZ Renormalization

The n_fth CWZ subscheme for renormalization of a bare partonic cross section is defined as follows: The $\overline{MS}$ scheme is used to subtract UV divergences from graphs with only AP's, and BPH (zero momentum subtraction) is used for any graphs with an NPQ. This removes all NPL's from all graphs. (This use of minimal and momentum subtraction was used in [78] to show that this causes a heavy quark to decouple completely order by order in the β function.) Explicitly, a_{0s}, the bare coupling, has the form

$$a_{0s} S_\epsilon \left(\frac{\mu_R^2}{\mu^2}\right)^{-\epsilon} = a_s^{\{n_f\}}(\mu_R^2) Z_{a_s}^{CWZ,n_f}\left(a_s^{\{n_f\}}(\mu_R^2,\epsilon),\epsilon\right), \tag{5.4.1}$$

where

$$Z_{a_s}^{CWZ,n_f}\left(a_s^{\{n_f\}}(\mu_R^2,\epsilon),\epsilon\right) = Z_{a_s}^{\text{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2,\epsilon),\mu_R^2,\epsilon\right) Z_{a_s}^{\overline{MS},6}\left(a_s^{\{6\}}(\mu_R^2,\epsilon),\epsilon\right). \tag{5.4.2}$$

$Z_{a_s}^{\overline{MS},6}$ is the usual renormalization constant in the $\overline{MS}$ scheme for 6 flavours, with the form given by equations (C.1.5) and (C.1.6), and removes all terms in a bare partonic cross section that are singular as $\epsilon \to 0$. It does not depend on any masses. $Z_{a_s}^{\text{mass},n_f}$ is non-singular as $\epsilon \to 0$, with the form

$$Z_{a_s}^{\text{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2,\epsilon),\mu_R^2,\epsilon\right) = 1 + \sum_{j=1}^{\infty} Z_{a_s}^{\text{mass},n_f,(j)}\left(\frac{m_{n_f+1}^2}{\mu_R^2},\ldots,\frac{m_6^2}{\mu_R^2},\epsilon\right) a_s^{\{n_f\}j}(\mu_R^2,\epsilon). \tag{5.4.3}$$

Here, we explicitly show the mass dependence of the $Z_{a_s}^{\text{mass},n_f(j)}$ in order to emphasize that they do not depend on AP masses, only NPQ masses. The purpose of $Z_{a_s}^{\text{mass},n_f}$ is to remove all NPL's, i.e. all terms of $O\left(\ln^j \frac{\mu_R^2}{m_{NPQ}^2}\right)$ (where $j \geq 0$) in bare partonic cross sections (and, more generally, in any off-shell Green function). Then, in the resulting bare partonic cross section, as all the NPQ masses go to infinity, they decouple since after renormalization any contributions from NPQ's are of $O\left(\frac{Q^2}{m_{NPQ}^2}\right)$ or less. After doing this, if we then send all the AP masses to zero, but leave m_i finite in terms of $O\left(\ln^j \frac{Q^2}{m_{AP}^2}\right)$, the bare partonic cross section reduces to the (mass regularized) one in a theory with n_f massless flavours only, renormalized in the $\overline{MS}$ scheme. $Z_{a_s}^{\text{mass},n_f}$ is such that $a_s^{\{n_f\}}$ evolves according to the β function for a theory with just n_f flavours when the $\overline{MS}$ scheme has been used for renormalization.

Note that $a_s^{\{6\}}$ in equation (5.4.2) can be written in terms of $a_s^{\{n_f\}}$, by using equations (5.4.1) and (5.4.2) to write

$$a_s^{\{6\}}(\mu_R^2,\epsilon) = a_s^{\{n_f\}}(\mu_R^2,\epsilon) Z_{a_s}^{\mathrm{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2,\epsilon),\mu_R^2,\epsilon\right). \tag{5.4.4}$$

In equation (5.4.4), we have used the fact that $Z_{a_s}^{\mathrm{mass},6} = 1$, since the 6th CWZ subscheme is just the $\overline{MS}$ scheme. $Z_{a_s}^{\mathrm{mass},n_f}$ is such that $a_s^{\{n_f\}}$ evolves according to the $\overline{MS}$ scheme in a theory with n_f flavours. We now show this explicitly at LO. Equation (5.4.4) to NLO reads

$$a_s^{\{6\}}(\mu_R^2,\epsilon) = a_s^{\{n_f\}}(\mu_R^2,\epsilon) + Z_{a_s}^{\mathrm{mass},n_f(1)}(\mu_R^2,\epsilon) a_s^{\{n_f\}2}(\mu_R^2,\epsilon). \tag{5.4.5}$$

Differentiating equation (5.4.5) with respect to $\ln\mu_R^2$ and using equation (C.1.8), then using equation (5.4.5) to write $a_s^{\{6\}}$ in terms of $a_s^{\{n_f\}}$, we find that

$$\frac{da_s^{\{n_f\}}(\mu_R^2,\epsilon)}{d\ln\mu_R^2} = -\epsilon a_s^{\{n_f\}}(\mu_R^2,\epsilon)$$
$$-\left(\epsilon Z_{a_s}^{\mathrm{mass},n_f(1)}(\mu_R^2,\epsilon) + \beta_0^{\{6\}} + \frac{dZ_{a_s}^{\mathrm{mass},n_f(1)}}{d\ln\mu_R^2}(\mu_R^2,\epsilon)\right) a_s^{\{n_f\}2}(\mu_R^2,\epsilon) + O(a_s^3). \tag{5.4.6}$$

Now [77]

$$Z_{a_s}^{\mathrm{mass},n_f(1)}(\mu_R^2,\epsilon) = \frac{1}{\epsilon}\frac{2}{3}T_R \sum_{I=n_f+1}^{6}\left(1-\left(\frac{m_I^2}{\mu_R^2}\right)^{\epsilon}\right), \tag{5.4.7}$$

so that

$$\epsilon Z_{a_s}^{\mathrm{mass},n_f(1)}(\mu_R^2,\epsilon) + \beta_0^{\{6\}} + \frac{dZ_{a_s}^{\mathrm{mass},n_f(1)}}{d\ln\mu_R^2}(\mu_R^2,\epsilon) = \beta_0^{\{n_f\}}, \tag{5.4.8}$$

where equation (2.4.5) has been used, and so equation (5.4.6) reads

$$\frac{da_s^{\{n_f\}}(\mu_R^2,\epsilon)}{d\ln\mu_R^2} = -\epsilon a_s^{\{n_f\}}(\mu_R^2,\epsilon) - \beta_0^{\{n_f\}} a_s^{\{n_f\}2}(\mu_R^2,\epsilon) + O(a_s^3), \tag{5.4.9}$$

which was what we set out to show. Note that after renormalization in the n_fth CWZ subscheme there are still AL's in the perturbation series, i.e. terms

of $O\left(\ln^j \frac{Q^2}{m_{AP}^2}\right)$. Recall that these large mass logarithms can only be removed via factorization, and not by a careful choice of $Z_{a_s}^{\text{mass},n_f}$. It is in anticipation of the fact that we intend to factorize that we do not choose to use any CWZ subscheme below the n_fth one when $O(m_{n_f}) < Q < O(m_{n_f+1})$. If we did, there would still be no NPL's, but new "pseudo AL's" arise, introduced by $Z_{a_s}^{\text{mass},\bar{n}_f}$, where $\bar{n}_f < n_f$, that are not present in the n_fth CWZ subscheme, and it would not be possible to factor these out. Furthermore, the n_fth CWZ subscheme will satisfy our requirements of section 5.3.

We take the limit $\epsilon \to 0$ for the rest of this chapter. Equation (5.4.5) is well defined in this limit, and reads

$$a_s^{\{6\}}(\mu_R^2) = a_s^{\{n_f\}}(\mu_R^2) Z_{a_s}^{\text{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2), \mu_R^2\right), \tag{5.4.10}$$

where

$$\begin{aligned} & Z_{a_s}^{\text{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2), \mu_R^2\right) \qquad (5.4.11) \\ &= 1 + \sum_{i=1}^{\infty} \sum_{j_{n_f+1}=0}^{\infty} \cdots \sum_{j_6=0}^{\infty} a_s^{\{n_f\}i}(\mu_R^2) Z_{a_s, j_{n_f+1} \cdots j_6}^{\text{mass},n_f,(i)} \prod_{k=n_f+1}^{6} \ln^{j_k} \frac{\mu_R^2}{m_k^2}. \end{aligned}$$

Note that $Z_{a_s, j_{n_f+1} \cdots j_6}^{\text{mass},n_f,(i)} = 0$ when $j_{n_f+1} + \ldots + j_6$ is sufficiently large. Furthermore $Z_{a_s, j_{n_f+1} \cdots j_6}^{\text{mass},n_f,(i)}$ is symmetric in the indices $j_{n_f+1}, \ldots, j_6$. Note that equation (5.4.10) is all that is needed to perform a change in renormalization scheme from the $\overline{MS}$ scheme to the n_fth CWZ subscheme.

The expansion of a bare partonic cross section $\widehat{S}_{2i}$ in $a_s^{\{n_f\}}$ will be written

$$\widehat{S}_{2i}(N-1, Q^2) = \sum_{j=0}^{\infty} a_s^{\{n_f\}j}(\mu_R^2) \widehat{S}_{2i}^{\{n_f\}(j)}\left(N-1, Q^2, \mu_R^2,\right). \tag{5.4.12}$$

The $Z_{a_s, j_{n_f+1}, \ldots, j_6}^{\text{mass},n_f,(i)}$ can be obtained by first renormalizing $\widehat{S}_{2i}$ in the $\overline{MS}$ scheme, then replacing all occurrences of $a_s^{\{6\}}$ with the right hand side of equation (5.4.10), and then demanding that $Z_{a_s}^{\text{mass},n_f}$ is such that as all NPQ masses go to infinity,

each term $\widehat{S}_{2i}^{\{n_f\}(j)}$ reduces to its $\overline{MS}$ form in a theory where there are only n_f flavours. This completely defines the n_fth CWZ subscheme for renormalization. Since the decoupling theorem applies in this scheme [75], the $\widehat{S}_{2\ AP}^{\{n_f\}(j)}$ are free of "cross-terms" of $O\left(f\left(\frac{m_{AP}^2}{Q^2}\right)\ln^k\frac{Q^2}{m_{NPQ}^2}\right)$ for $k \geq 0$, even though we have not introduced any AP mass dependence into $Z_{a_s}^{\text{mass},n_f}$.

We will define the thresholds for renormalization to be $m_i' = O(m_i)$, i.e. when $m_{n_f}' \leq \mu_R < m_{n_f+1}'$, we will use the n_fth CWZ subscheme. The exact value of m_i' does not matter, as long as $\ln^n \frac{m_i'}{m_i} = O(1)$. In other words, in the threshold region it does not matter whether we remove the NPL's (or, more correctly put, absorb them into the coupling constant) since they are not large here.

We will interpret $a_s^{\{n_f\}}(\bar{\mu}^2)$ in the case $\bar{\mu} \neq \mu_R$ in a perturbation series as

$$a_s^{\{n_f\}}(\bar{\mu}^2) = h_\infty^{\{n_f\}}\left(\frac{\bar{\mu}^2}{\mu_R^2}, a_s^{\{n_f\}}(\mu_R^2)\right), \tag{5.4.13}$$

where $h_\infty^{\{n_f\}}$ is that in equation (2.18.8) for $n = \infty$, and in a theory with only n_f quarks where the $\overline{MS}$ scheme has been used for renormalization.

If the $\widehat{S}_{2i}^{\{n_f-1\}(i)}\left(N-1, Q^2, \mu_R^2, \{m_j^2\},\right)$ are known, it is trivial to obtain the $\widehat{S}_{2i}^{\{n_f\}(i)}\left(N-1, Q^2, \mu_R^2, \{m_j^2\},\right)$. However, the relation between $a_s^{\{n_f\}}(\mu_R^2)$ and $a_s^{\{n_f-1\}}(\mu_R^2)$ must be known before a change of subscheme can be done. This can be found by taking the result which follows from equation (5.4.10),

$$a_s^{\{n_f\}}(\mu_R^2) Z_{a_s}^{\text{mass},n_f}\left(a_s^{\{n_f\}}(\mu_R^2), \mu_R^2\right) = a_s^{\{n_f-1\}}(\mu_R^2) Z_{a_s}^{\text{mass},n_f-1}\left(a_s^{\{n_f-1\}}(\mu_R^2), \mu_R^2\right), \tag{5.4.14}$$

and then solving perturbatively for $a_s^{\{n_f\}}(\mu_R^2)$ to get

$$a_s^{\{n_f\}}(\mu_R^2) = a_s^{\{n_f-1\}}(\mu_R^2) A_{a_s}^{\{n_f\}}\left(a_s^{\{n_f-1\}}(\mu_R^2), \ln\frac{\mu_R^2}{m_{n_f}^2}\right). \tag{5.4.15}$$

Note that $A_{a_s}^{\{n_f\}}$ contains terms of $O\left(\ln^j \frac{\mu_R^2}{m_{n_f}^2}\right)$, but no other logarithms. This

is because, for $O(m_{n_f-1}) < Q < O(m_{n_f})$, the $\widehat{S}_{2i}^{\{n_f-1\}(j)}$ of equation (5.4.12) are free of NPL's, but $\widehat{S}_{2i}^{\{n_f\}(j)}$ contains terms of $O\left(\ln^j \frac{\mu_R^2}{m_{n_f}^2}\right)$. Note that $A_{a_s}^{\{n_f\}}$ can be perturbatively calculated when $\mu_R = O(Q)$. Then $a_s^{\{n_f\}}(\mu_R^2)$ can be evolved from $\mu_R^2 = m_{n_f}'^2$ using the β equation, equation (2.4.2), for n_f flavours. Equation (5.4.15) has been calculated to N^{3}LO in [79]. The result to NNLO is

$$a_s^{\{n_f\}}(\mu_R^2) = a_s^{\{n_f-1\}}(\mu_R^2)\left(1 + a_s^{\{n_f-1\}}(\mu_R^2)\frac{1}{3}\ln\frac{\mu_R^2}{m_{n_f}^2}\right.$$

$$\left. + a_s^{\{n_f-1\}2}(\mu_R^2)\left(\frac{1}{9}\ln^2\frac{\mu_R^2}{m_{n_f}^2} + \frac{19}{6}\ln\frac{\mu_R^2}{m_{n_f}^2} + \frac{7}{6}\right)\right) + O(a_s^4). \qquad (5.4.16)$$

Note that a_s is then not continuous across threshold in general, even if we set $m'_{n_f} = m_{n_f}$. In fact, it is preferable to rearrange this equation using equation (2.4.9) to write $\Lambda_{QCD}^{\{n_f\}}$ in terms of $\Lambda_{QCD}^{\{n_f-1\}}$, and then use equation (2.4.9) for the functional dependence of a_s on μ_R^2.

At NLO, no large mass logarithms other than AL's occur. One may wonder therefore whether it matters what renormalization subscheme we use. However, after factorization, a bad choice for n_f in renormalization will result in a truncated series that may appear convergent (or at least free of large mass logarithms), but where the neglected higher order terms are in fact larger than the NLO term due to unremoved large mass logarithms, so that the truncated series is part of a non converging series. However, again the difference would still be of $O(a_s^2)$. What really matters is that our NLO expressions in this chapter for all the quantities that make up F_2 are such that in future one could simply add higher order corrections to our expressions, without having to change our expressions.

5.5 CWZ Factorization

One result of introducing NPQ's into our theory is the occurrence of NPQ initiated bare partonic cross sections $\widehat{S}_{2\ NPQ}$, which have no ZM-VFNS analogue. These quantities are zero as NPQ masses go to infinity, because of the $\theta\left(1 - \chi_{I,\alpha}\frac{\eta}{\xi}\right)$ factor. (Note if the initial parton is I, there will always be a parton I in the final

state so that $\hat{s}_{\text{th}I,\alpha} = m_I^2 +$ positive terms.) But we do not wish to take this limit. In that case the AL's in the $\widehat{S}_{2\ NPQ}$ are suppressed by the step functions, and so, although they still need to be dealt with, these potential singularities are a different kind to the AL's in the $\widehat{S}_{2\ AP}$ and so can be factorized out separately. Thus we can and will deal with the $\widehat{S}_{2\ AP}$ separately from the $\widehat{S}_{2\ NPQ}$. For now we note that $\sum_{I=n_f+1}^{6} \widehat{S}_{2I} f_{0I}^{+}$ is finite as AP masses go to zero and NPQ masses go to infinity, since the potential mass singularities are of a different nature to those in $\sum_{i=0}^{n_f} \widehat{S}_{2i} f_{0i}^{+}$. In fact we will see later that the contribution of $\sum_{I=n_f+1}^{6} \widehat{S}_{2I} f_{0I}^{+}$ to F_2^{Norm} is small.

We will omit parton indices unless they are essential for clarity. Thus, for example, f_{0i}^{+} for $i = 0, ..., 6$ will be written as f_0^{+}, which is a 7 component vector. We put the superscript "$[n_f]$" on factorization scheme dependent quantities to show that we are using the n_fth CWZ factorization subscheme. Note the n_fth CWZ subscheme for factorization must not be confused with the n_fth CWZ subscheme for renormalization, although there are similarities. Note that we are putting the superscript "$\{n_f\}$" on a quantity to indicate that the n_fth CWZ subscheme for renormalization has been used and that this quantity depends on the value of n_f. We are using the superscript "$[n_f]$" to do the same job for factorization. Explicitly, as in the massless case, the factorization becomes

$$\widetilde{S}_2^{[n_f]}(N-1, Q^2, \mu_F^2) = \widehat{S}_2(N-1, Q^2)\Gamma^{[n_f]-1}(N, \mu_F^2) \tag{5.5.1}$$

$$f^{[n_f]+}(N, \mu_F^2) = \Gamma^{[n_f]}(N, \mu_F^2) f_0^{+}(N), \tag{5.5.2}$$

so that

$$F_2^{Norm}(N-1, Q^2) = \widetilde{S}_2^{[n_f]}(N-1, Q^2, \mu_F^2) f^{[n_f]+}(N, \mu_F^2). \tag{5.5.3}$$

Since there is no mixing of potential mass singularities between the NPQ and AP components of $\widehat{S}_{2i}$, as discussed above, and since we are not interested in factorizing NPQ initiated bare cross sections at present, $\Gamma^{[n_f]}$ is of the form

$$\Gamma^{[n_f]} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \Gamma^{[n_f]}_{AP\ AP} \end{pmatrix}, \tag{5.5.4}$$

where $\mathbb{1}$ is the $(6-n_f)\times(6-n_f)$ unit matrix which acts only on NPQ's, and $\Gamma^{[n_f]}_{AP\ AP}$ is an $(n_f+1)\times(n_f+1)$ matrix that acts only on AP's. The AP components of $\Gamma^{[n_f]}$, $\Gamma^{[n_f]}_{AP\ AP}$ can be defined to be any suitable Γ of a theory with n_f massless quarks in which all partons are given a small mass to regularize mass divergences, and so there is no need for $\Gamma^{[n_f]}$ to depend on NPQ masses. Then

$$\Gamma^{[n_f]}(N,\mu_F^2) = \mathbb{1} + \sum_{i=1}^{\infty}\sum_{j_0=0}^{\infty}\cdots\sum_{j_{n_f}=0}^{\infty} a_s^{\{n_f\}i}(\mu_F^2)\Gamma^{[n_f]\{n_f\}(i)}_{j_0\ldots j_{n_f}}(N)\prod_{k=0}^{n_f}\ln^{j_k}\frac{\mu_F^2}{m_k^2}, \tag{5.5.5}$$

where $\Gamma^{[n_f]\{n_f\}(i)}_{j_0\ldots j_{n_f}} = 0$ for $j_0+\ldots+j_{n_f} > i$. In the CWZ subscheme for factorization, this is the form that $\Gamma^{[n_f]}$ takes. A heuristic justification for this, using the results of [9], is as follows: Firstly, K as defined in equation (2.8.21) is non-singular in the limits $m_{NPQ}\to\infty$ after renormalizing in the n_fth CWZ subscheme, because of the decoupling theorem. Thus $\Gamma^{[n_f]}$ as defined in equation (2.8.19) must be free of cross-terms of $O\left(f\left(\frac{m_{AP}^2}{\mu_F^2}\right)\ln^j\frac{\mu_F^2}{m_{NPQ}^2}\right)$ for $j\geq 1$. However, $\Gamma^{[n_f]}$ may still contain terms of $O\left(\left(\frac{\mu_F^2}{m_{NPQ}^2}\right)^l\ln^j\frac{\mu_F^2}{m_{AP}^2}\right)$ for $l\geq 1$ and $j\geq 0$. In the n_fth CWZ subscheme for renormalization, the f_{AP}^+ as defined in (2.10.7) evolve according to a theory with just n_f flavours where the $\overline{MS}$ scheme has been used. Thus $\Gamma^{[n_f]}_{AP\ AP}$ must evolve the same way, i.e.

$$\frac{d}{d\ln\mu_F^2}\Gamma^{[n_f]}\left(N,\mu_F^2\right) = a_s^{\{n_f\}}(\mu_F^2)P^{[n_f]}(N,a_s^{\{n_f\}}(\mu_F^2))\Gamma^{[n_f]}\left(N,\mu_F^2\right), \tag{5.5.6}$$

where $P^{[n_f]}$ is a matrix containing the mass independent $\overline{MS}$ splitting functions in a theory with just n_f flavours where the $\overline{MS}$ scheme has been used. (See below for the explicit matrix form of $P^{[n_f]}$.) Solving this equation order by order shows there cannot be terms of $O(\mu_F^l)$ in $\Gamma^{[n_f]}$, and thus $\Gamma^{[n_f]}$ is free of terms of

$O\left(\left(\frac{\mu_F^2}{m_{NPQ}^2}\right)^l \ln^j \frac{\mu_F^2}{m_{AP}^2}\right)$ for $l \geq 1$ and $j \geq 0$. Thus $\Gamma^{[n_f]}$ has the form in equation (5.5.5), as we set out to show. From [9], the $\widetilde{S}_{2\ AP}^{[n_f]}$ are non singular as AP masses go to zero and NPQ masses go to infinity. Furthermore, as the AP masses go to zero and the NPQ masses go to infinity, each term in the perturbation series for $\widetilde{S}_{2\ AP}^{[n_f]}$ reduces to those for the $\overline{MS}$ $\widetilde{F}_{2\ AP}$ for n_f massless flavours.

Some of the components of $\Gamma^{[n_f]}$ have been calculated to as high as NNLO [80, 81, 82]. One method of calculating the $\Gamma_{j_0 \cdots j_{n_f}}^{[n_f]\{n_f\}(i)}(N)$ of equation (5.5.5) would be to demand that as NPQ masses go to infinity and AP masses go to zero, the factorized partonic cross sections of any process reduce to their $\overline{MS}$ form for n_f massless flavours. This completely defines the n_fth CWZ subscheme for factorization. Indeed, we will see later that $\Gamma_{Ig}^{[n_f](1)}$ can be found from the bare NLO DIS gluon initiated cross section in this way. Since the $\widetilde{S}_{2\ AP}^{[n_f]}$ are well defined as AP masses go to zero [9], they are free of terms of $O\left(\left(\frac{Q^2}{m_{NPQ}^2}\right)^j \ln^k \frac{Q^2}{m_{AP}^2}\right)$, even though we have not introduced any NPQ mass dependence into $\Gamma^{[n_f]}$.

Note that, when $Q = O(m_i)$, we can choose the value of n_f that labels the CWZ subscheme used for factorization to either i or $i+1$, since $\ln^k \frac{Q^2}{m_i^2}$ is small, so these logarithms do not need resumming. The same applies to the n_f that labels the CWZ subscheme used for renormalization. In fact, these two definitions of n_f need not be made to be equal to one another when $Q = O(m_i)$ - we could have one equal to i and the other equal to $i+1$, for example.

The NPQ PDF's do not evolve with μ_F^2 and do not mix with the AP's. They are given by their bare PDF's f_{0i}.

We can now show the explicit component form of some of the quantities mentioned above. We define the following notation:

$$f_0^+ = \left(f_{0NPQ}^+, f_{0n_f}^+, f_{0AP}^+\right). \tag{5.5.7}$$

Here, f_{0NPQ}^+ is a vector composed of all those f_{0i}^+ with $i > n_f$, and f_{0AP}^+ is a vector composed of all those f_{0i}^+ with $i < n_f$. In this notation, equation (5.5.2) becomes

$$f^{[n_f]+} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & \Gamma^{[n_f]}_{n_f n_f} & \Gamma^{[n_f]}_{n_f\ AP} \\ 0 & \Gamma^{[n_f]}_{AP\ n_f} & \Gamma^{[n_f]}_{AP\ AP} \end{pmatrix} \begin{pmatrix} f^+_{0NPQ} \\ f^+_{0n_f} \\ f^+_{0AP} \end{pmatrix}. \tag{5.5.8}$$

Here, the n_fth quark is an AP. Considering now the $(n_f - 1)$th CWZ subscheme for both factorization and renormalization,

$$f^{[n_f-1]+} = \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \Gamma^{[n_f-1]}_{AP\ AP} \end{pmatrix} \begin{pmatrix} f^+_{0NPQ} \\ f^+_{0n_f} \\ f^+_{0AP} \end{pmatrix}. \tag{5.5.9}$$

In this case the n_fth quark is now an NPQ. The splitting functions are of the form

$$a_s^{\{n_f\}} P^{[n_f]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_s^{\{n_f\}} P^{[n_f]}_{n_f n_f} & a_s^{\{n_f\}} P^{[n_f]}_{n_f\ AP} \\ 0 & a_s^{\{n_f\}} P^{[n_f]}_{AP\ n_f} & a_s^{\{n_f\}} P^{[n_f]}_{AP\ AP} \end{pmatrix} \tag{5.5.10}$$

$$a_s^{\{n_f-1\}} P^{[n_f-1]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_s^{\{n_f-1\}} P^{[n_f-1]}_{AP\ AP} \end{pmatrix}. \tag{5.5.11}$$

We now consider the procedure for the calculation of the $\widetilde{S}^{[n_f](i)}_{2i}$. We expand equation (5.5.1), choosing the n_fth CWZ renormalization as well as factorization subscheme. Then, matching coefficients of $a_s^{\{n_f\}}$, we find

$$\widetilde{S}^{[n_f]\{n_f\}(i)}_2 = \widehat{S}^{\{n_f\}(i)}_2 - \sum_{j=0}^{i-1} \widetilde{S}^{[n_f]\{n_f\}(j)}_2 \Gamma^{[n_f]\{n_f\}(i-j)}. \tag{5.5.12}$$

Thus to LO we have

$$\widetilde{S}_{2I}^{[n_f]\{n_f\}(0)} = \widehat{S}_{2I}^{\{n_f\}(0)}. \tag{5.5.13}$$

Note that $\widehat{S}_{2g}^{\{n_f\}(0)} = 0$, so therefore $\widetilde{S}_{2g}^{[n_f]\{n_f\}(0)} = 0$. To NLO, we have

$$\widetilde{S}_{2I}^{[n_f]\{n_f\}(1)} = \widehat{S}_{2I}^{\{n_f\}(1)} - \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(0)} \Gamma_{KI}^{[n_f]\{n_f\}(1)} \tag{5.5.14}$$

$$\widetilde{S}_{2g}^{[n_f]\{n_f\}(1)} = \widehat{S}_{2g}^{\{n_f\}(1)} - \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(0)} \Gamma_{Kg}^{[n_f]\{n_f\}(1)}. \tag{5.5.15}$$

Equation (5.5.15) was first suggested in [76], except that factorized quark PDF's were neglected. This subtraction of large mass logarithms at NLO is well known, and is called the *ACOT scheme*. To NNLO,

$$\begin{aligned}\widetilde{S}_{2I}^{[n_f]\{n_f\}(2)} = \widehat{S}_{2I}^{\{n_f\}(2)} &- \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(1)} \Gamma_{KI}^{[n_f]\{n_f\}(1)} - \widetilde{S}_{2g}^{[n_f]\{n_f\}(1)} \Gamma_{gI}^{[n_f]\{n_f\}(1)} \\ &- \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(0)} \Gamma_{KI}^{[n_f]\{n_f\}(2)}\end{aligned} \tag{5.5.16}$$

$$\begin{aligned}\widetilde{S}_{2g}^{[n_f]\{n_f\}(2)} = \widehat{S}_{2g}^{[n_f](2)} &- \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(1)} \Gamma_{Kg}^{[n_f]\{n_f\}(1)} - \widetilde{S}_{2g}^{[n_f]\{n_f\}(1)} \Gamma_{gg}^{[n_f]\{n_f\}(1)} \\ &- \sum_{K=1}^{n_f} \widetilde{S}_{2K}^{[n_f]\{n_f\}(0)} \Gamma_{Kg}^{[n_f]\{n_f\}(2)}.\end{aligned} \tag{5.5.17}$$

We will define the thresholds for factorization to be $m_i'' = O(m_i)$, i.e. when $m_{n_f}'' \leq \mu_F < m_{n_f+1}''$, we will use the n_fth CWZ subscheme. Just as for renormalization thresholds earlier, the exact value of m_i'' does not matter, as long as $\ln^n \frac{m_i''}{m_i} = O(1)$. In the threshold region it does not matter whether we remove the AL's (i.e. absorb them into the PDF's), since they are not large here.

5.6 Matching

In this section we consider the change from one CWZ factorization subscheme to another. From equation (5.5.2), since the f_{0i}^{+} are scheme independent, we find the relation between the factorized PDF's in two neighbouring CWZ subschemes is given by

$$f^{[n_f]+}(N,\mu_F^2) = A^{[n_f]}(N,\mu_F^2) f^{[n_f-1]+}(N,\mu_F^2), \tag{5.6.1}$$

where

$$A^{[n_f]}(N,\mu_F^2) = \Gamma^{[n_f]}(N,\mu_F^2)\left(\Gamma^{[n_f-1]}(N,\mu_F^2)\right)^{-1}. \tag{5.6.2}$$

At threshold, i.e. when $\mu_F^2 = m_{n_f}''^2$, the matching which needs to be done here takes the form

$$f^{[n_f]+}(N,m_{n_f}''^2) = A^{[n_f]}(N,m_{n_f}''^2) f^{[n_f-1]+}(N,m_{n_f}''^2). \tag{5.6.3}$$

Likewise, for the matching of the $\widetilde{S}_{2i}^{[n_f]}$, from equation (5.5.1) we must have

$$\widetilde{S}_2^{[n_f]}(N-1,Q^2,\mu_F^2) = \widetilde{S}_2^{[n_f-1]}(N-1,Q^2,\mu_F^2) A^{[n_f]-1}(N,\mu_F^2). \tag{5.6.4}$$

Equation (5.6.4) shows that $A^{[n_f]}$ is free of terms of $O\left(\ln^n \frac{\mu_F^2}{m_i^2}\right)$ for $i \neq n_f$, since, for $O(m_{n_f}^2) < \mu_F^2 < O(m_{n_f+1}^2)$, the only large mass logarithms in $\widetilde{S}_2^{[n_f-1]}(N-1,Q^2,\mu_F^2)$ are the ones of $O\left(\ln^n \frac{\mu_F^2}{m_{n_f}^2}\right)$, whereas $\widetilde{S}_2^{[n_f-1]}(N-1,Q^2,\mu_F^2)$ contains no large mass logarithms. Note from the definition in equation (5.6.2) that $A^{[n_f]}$ contains no terms of $O\left(\left(\frac{m_i^2}{\mu_F^2}\right)^m\right)$ or $O\left(\left(\frac{\mu_F^2}{m_i^2}\right)^m\right)$, for $i = 0,...,6$. Note that, over the whole region $O(m_{n_f}^2) < \mu_F^2 < O(m_{n_f+1}^2)$, the perturbation series on the right hand side of equation (5.6.4) does not become less convergent as AP masses go to zero and NPQ masses go to infinity, since $\widetilde{S}_2^{[n_f]}$ is free of large mass logarithms in this region.

Thus $A^{[n_f]}$ only contains terms of $O\left(\ln^n \frac{\mu_F^2}{m_{n_f}^2}\right)$. Therefore $A^{[n_f]}$ can be perturbatively evaluated when $\mu_F^2 = O(m_{n_f}^2)$ (but it cannot be perturbatively evaluated for any other order of magnitude of μ_F). Thus equation (5.6.3) can be obtained perturbatively. Then the new $f^{[n_f]+}$ can be evolved from $\mu_F = m''_{n_f}$ using the DGLAP equation for n_f flavours in the $\overline{MS}$ scheme.

To NNLO,

$$A^{[n_f]}(N, m''^2_{n_f}) = \mathbb{1} + a_s^{\{n_f\}}(m''^2_{n_f}) \left[\Gamma^{[n_f]\{n_f\}(1)}(N, m''^2_{n_f}) - \Gamma^{[n_f-1]\{n_f\}(1)}(N, m''^2_{n_f})\right]$$
$$a_s^{\{n_f\}2}(m''^2_{n_f}) \Big[\Gamma^{[n_f]\{n_f\}(2)}(N, m''^2_{n_f}) - \Gamma^{[n_f-1]\{n_f\}(2)}(N, m''^2_{n_f})$$
$$+\Gamma^{[n_f-1]\{n_f\}(1)}(N, m''^2_{n_f}) \left(\Gamma^{[n_f-1]\{n_f\}(1)}(N, m''^2_{n_f}) - \Gamma^{[n_f]\{n_f\}(1)}(N, m''^2_{n_f})\right)\Big]. \quad (5.6.5)$$

To show the form of $A^{[n_f]}$, we write matrices M_{ij} in a more expanded form than before:

$$M_{ij} = \begin{pmatrix} i & > & n_f \\ j \quad i, j = n_f & i = n_f,\ j < n_f & i = n_f,\ j = g \\ > \quad i < n_f,\ j = n_f & i, j < n_f & i < n_f,\ j = g \\ n_f \quad i = g,\ j = n_f & i = g,\ j < n_f & i, j = g \end{pmatrix}. \quad (5.6.6)$$

Then to NLO, omitting the "$\{n_f\}$" superscript for brevity, we have

$$A^{[n_f]} =$$

$$\begin{pmatrix} \mathbb{1} & 0 & 0 & 0 \\ 0 & 1+a_s\Gamma^{[n_f](1)}_{n_f\ n_f} & a_s\Gamma^{[n_f](1)}_{n_f\ J} & a_s\Gamma^{[n_f](1)}_{n_f\ g} \\ 0 & a_s\Gamma^{[n_f](1)}_{I\ n_f} & \delta_{IJ}+a_s\left(\Gamma^{[n_f](1)}_{I\ J}-\Gamma^{[n_f-1](1)}_{I\ J}\right) & a_s\left(\Gamma^{[n_f](1)}_{I\ g}-\Gamma^{[n_f-1](1)}_{I\ g}\right) \\ 0 & a_s\Gamma^{[n_f](1)}_{g\ n_f} & a_s\left(\Gamma^{[n_f](1)}_{g\ J}-\Gamma^{[n_f-1](1)}_{g\ J}\right) & 1+a_s\left(\Gamma^{[n_f](1)}_{g\ g}-\Gamma^{[n_f-1](1)}_{g\ g}\right) \end{pmatrix}. \quad (5.6.7)$$

The elements of the matrix in equation (5.6.7) are presented in [82], where

$$A_{n_f n_f}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) = \Gamma_{n_f n_f}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) = P_{qq}^{(0)}(N)\ln\frac{\mu_F^2}{m_{n_f}^2} \tag{5.6.8}$$

$$A_{n_f g}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) = \Gamma_{n_f g}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) = 2P_{qg}^{(0)}(N)\ln\frac{\mu_F^2}{m_{n_f}^2} \tag{5.6.9}$$

$$\begin{aligned} A_{gg}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) &= \Gamma_{gg}^{[n_f]\{n_f\}(1)}(N,\mu_F^2) - \Gamma_{gg}^{[n_f-1]\{n_f\}(1)}(N,\mu_F^2) \\ &= \left(P_{gg}^{[n_f](0)} - P_{gg}^{[n_f-1](0)}\right)\ln\frac{\mu_F^2}{m_{n_f}^2} = -\frac{2}{3}T_R\ln\frac{\mu_F^2}{m_{n_f}^2}, \end{aligned} \tag{5.6.10}$$

and all the other $A_{ij}^{[n_f]\{n_f\}(1)}(N,\mu_F^2)$ are zero. Equation (5.6.3) therefore gives the matching conditions to NLO as

$$f_{NPQ}^{[n_f]+}(N,m_{n_f}''^2) = f_{NPQ}^{[n_f-1]+}(N,m_{n_f}''^2) \tag{5.6.11}$$

$$\begin{aligned} f_{n_f}^{[n_f]+}(N,m_{n_f}''^2) &= \left(1 + a_s^{\{n_f\}}(m_{n_f}''^2)\Gamma_{n_f n_f}^{[n_f]\{n_f\}(1)}(N,m_{n_f}''^2)\right) f_{n_f}^{[n_f-1]+}(N,m_{n_f}''^2) \\ &+ a_s^{\{n_f\}}(m_{n_f}''^2)\Gamma_{n_f g}^{[n_f]\{n_f\}(1)}(N,m_{n_f}''^2) f_g^{[n_f-1]+}(N,m_{n_f}''^2) \end{aligned} \tag{5.6.12}$$

$$f_I^{[n_f]+}(N,m_{n_f}''^2) = f_I^{[n_f-1]+}(N,m_{n_f}''^2) \quad \text{for} \quad I < n_f \tag{5.6.13}$$

$$\begin{aligned} &f_g^{[n_f]+}(N,m_{n_f}''^2) \\ &= \left(1 + a_s^{\{n_f\}}(m_{n_f}''^2)\left(\Gamma_{gg}^{[n_f]\{n_f\}(1)}(N,m_{n_f}''^2) - \Gamma_{gg}^{[n_f-1]\{n_f\}(1)}(N,m_{n_f}''^2)\right)\right) \\ &\qquad \times f_g^{[n_f-1]+}(N,m_{n_f}''^2). \end{aligned} \tag{5.6.14}$$

Defining

$$\Sigma^{[n_f]} = \sum_{I=1}^{n_f} f_I^{[n_f]+} \qquad T_{n^2-1}^{[n_f]} = \sum_{I=1}^{n} f_I^{[n_f]+} - n f_n^{[n_f]+}, \tag{5.6.15}$$

then the matching conditions, equations (5.6.11) to (5.6.13), become

$$\Sigma^{[n_f]}(N, m''^2_{n_f}) = \Sigma^{[n_f-1]}(N, m''^2_{n_f}) + a_s^{\{n_f\}}(m''^2_{n_f}) 2P_{qg}^{(0)}(N) \ln \frac{m''^2_{n_f}}{m^2_{n_f}} f_g^{[n_f-1]+}(N, m''^2_{n_f}) \tag{5.6.16}$$

$$T^{[n_f]}_{n^2-1}(N, m''^2_{n_f}) = T^{[n_f-1]}_{n^2-1}(N, m''^2_{n_f}) \quad \text{for} \quad n = 2, ..., n_f - 1 \tag{5.6.17}$$

$$T^{[n_f]}_{n_f^2-1}(N, m''^2_{n_f}) = \Sigma^{[n_f-1]}(N, m''^2_{n_f})$$
$$-(n_f - 1) a_s^{\{n_f\}}(m''^2_{n_f}) 2P_{qg}^{(0)}(N) \ln \frac{m''^2_{n_f}}{m^2_{n_f}} f_g^{[n_f-1]+}(N, m''^2_{n_f}). \tag{5.6.18}$$

Note that, if we are working to $O(a_s^n)$, then F_2 has a discontinuity of $O(a_s^{n+1})$ at threshold. This is in contrast to [83, 84], where matching at threshold is *defined* such that F_2 is continuous and differentiable in $\ln Q^2$ across threshold. This is due to the neglect of the intrinsic (bare) heavy quark PDF's from the outset since they are small, so that $A^{[n_f]}$ has extra degrees of freedom requiring additional constraints. Indeed, we intend in section 5.7 to neglect the heavy quark PDF's, but here we have considered the more general case first, where heavy quarks are not neglected.

5.7 A Simplification Of The VFNS

For brevity, in this section we omit the "+" superscript on f_{0i}^+, the "2" subscript on $\widehat{S}_{2i}$ and the Mellin variable "N" in arguments. For simplicity, we will first consider the case of QCD with just one species of heavy quark H and the gluon g. By applying the same power counting arguments to equation (2.5.22) as we did to equation (2.5.18), we find that for a light hadron (i.e. one which is not a bound state of heavy quarks),

$$\widehat{S}_H f_{0H} = C_0^{\Theta=2} \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0} \cdot \hat{O}_H \cdot \hat{O}_H \cdot \frac{\mathbb{1}}{\mathbb{1} - K_0 \cdot (\mathbb{1} - \hat{O})} \cdot T_0.$$

$$= O\left(\frac{\Lambda^2_{QCD}}{m^2_H}\right), \tag{5.7.1}$$

where $\hat{O}_H$ is the H component of $\hat{O}$. Note that the above equation is well defined when the UV regulator is removed. Thus when the error on the theoretical calculation of a cross section is no less than of $O\left(\frac{\Lambda^2_{QCD}}{m^2_H}\right)$, the $\widehat{S}_H f_{0H}$ contribution to F can be neglected. We now examine a simplification of the VFNS that can be applied when this is done. Then setting $\widehat{S}_H f_{0H} = 0$, equation (5.1.16) becomes simply

$$F = \widehat{S}_g f_{0g}. \tag{5.7.2}$$

Since $\widehat{S}_H$ does not appear in equation (5.7.2), it is unnecessary to calculate it, even when m_H is neglected (or rather used only as a regulator of mass divergences), at least at the bare level and to all orders. However, when we come to perform actual calculations, in which we work only with truncated expressions, equation (5.7.2) is not suitable since $\widehat{S}_g$ is not a convergent series, so we would not use equation (5.7.2) as it stands. We would instead reshuffle all large mass logarithms between $\widehat{S}_g$ and f_{0g} via factorization to get a form of F that can be used for phenomenology. In particular, in the case $Q^2 > O(m^2_H)$, where parton H becomes active, we would factorize using the $n_f = 1$ CWZ subscheme, where equation (5.7.2) becomes

$$F = \widetilde{S}_H f_H + \widetilde{S}_g f_g, \tag{5.7.3}$$

where, from equation (5.5.2), we may choose

$$f_H = \Gamma_{Hg} f_{0g} \tag{5.7.4}$$

$$f_g = \Gamma_{gg} f_{0g}, \tag{5.7.5}$$

and, from equation (5.5.1),

$$\widetilde{S}_H = \widehat{S}_H(\Gamma^{-1})_{HH} + \widehat{S}_g(\Gamma^{-1})_{gH} \tag{5.7.6}$$

$$\widetilde{S}_g = \widehat{S}_H(\Gamma^{-1})_{Hg} + \widehat{S}_g(\Gamma^{-1})_{gg}. \tag{5.7.7}$$

Now, to all orders, F is invariant with respect to changes in $\widehat{S}_H$ in equations (5.7.6) and (5.7.7): Suppose we take $\widehat{S}_H \to \widehat{S}_H + \delta\widehat{S}_H$ in equations (5.7.6) and (5.7.7), so that

$$\widetilde{S}_H \to \widetilde{S}_H + \delta\widehat{S}_H(\Gamma^{-1})_{HH} \qquad \widetilde{S}_g \to \widetilde{S}_g + \delta\widehat{S}_H(\Gamma^{-1})_{Hg}, \tag{5.7.8}$$

and therefore from equation (5.7.3),

$$F \to F + \delta\widehat{S}_H(\Gamma^{-1})_{HH}f_H + \delta\widehat{S}_H(\Gamma^{-1})_{Hg}f_g. \tag{5.7.9}$$

From equations (5.7.4) and (5.7.5), we find, explicitly inverting the matrix Γ, that

$$(\Gamma^{-1})_{HH}f_H + (\Gamma^{-1})_{Hg}f_g = 0, \tag{5.7.10}$$

and so the change in F in equation (5.7.9) is zero.

However, the requirement for the cancellation of large mass logarithms between the $\widehat{S}_i$ and the Γ_{ij} at each order in equations (5.7.6) and (5.7.7), that is necessary for ensuring that the series for $\widetilde{S}_H$ and $\widetilde{S}_g$ are non-singular as $m_g^2, m_H^2 \to 0$, means that there are some restrictions on what we can choose for $\widehat{S}_H$ if we wish to obtain an accurate value for F, calculated in the form of equation (5.7.3). However, since $\widehat{S}_H$ is more complicated when the mass m_H is fully included than when m_H is used as just as a regulator of collinear divergences, it is beneficial to see what freedom we have to replace $\widehat{S}_H$ in equations (5.7.6) and (5.7.7) with something simpler.

The function we will choose to replace $\widehat{S}_H$ with, which we will call $\widehat{V}_H$, must be such that any terms of $O\left(\ln^n \frac{Q^2}{m_i^2}\right)$ as $m_i^2 \to 0$ for $i = H$ and $i = g$ in the

first terms of equations (5.7.6) and (5.7.7), that is $\widehat{S}_H(\Gamma^{-1})_{Hi}$ for $i = H$ and g respectively, are identical to the ones in $\widehat{V}_H(\Gamma^{-1})_{Hi}$. Let us examine what this means more explicitly. We set $\mu_R^2 = k_R Q^2$, but we will not show the k_R dependence in any arguments for brevity. Furthermore we omit the Mellin N in all arguments for brevity. Since $Q > O(m_H), O(m_g)$, the $\widetilde{S}_i$ must be such that they can be meaningfully expanded in powers of $\frac{m_H^2}{Q^2}$ and $\frac{m_g^2}{Q^2}$, To achieve this, we will need to expand the quantities used to construct the $\widetilde{S}_i$, e.g. $\widehat{S}_H$, in the same way, even if they are singular as $m_H, m_g \to 0$. Thus we write

$$\widehat{S}_H = \sum_{j_0=0}^{\infty} \sum_{j_1=0}^{\infty} \left(\frac{m_g^2}{Q^2}\right)^{j_0} \left(\frac{m_H^2}{Q^2}\right)^{j_1} A_{j_0 j_1}\left(\ln\frac{Q^2}{m_H^2}, \ln\frac{Q^2}{m_g^2}\right). \tag{5.7.11}$$

The $A_{j_0 j_1}$ are to be considered as Taylor series in their arguments. We find that the large mass logarithms of $\widehat{S}_H(\Gamma^{-1})_{Hi}$ are identical to the large mass logarithms of $A_{00}(\Gamma^{-1})_{Hi}$, since

$$\lim_{m_i \to 0} \left(\frac{m_i^2}{Q^2}\right)^n \ln^m \frac{Q^2}{m_i^2} \to 0 \tag{5.7.12}$$

for $n \geq 1$. This means that we are free to replace $\widehat{S}_H$ in equations (5.7.6) and (5.7.7) with $\widehat{V}_H$, where

$$\widehat{V}_H = A_{00}\left(\ln\frac{Q^2}{m_H^2}, \ln\frac{Q^2}{m_g^2}\right) + R\left(\frac{m_H^2}{Q^2}, \frac{m_g^2}{Q^2}\right) \tag{5.7.13}$$

and R obeys the condition

$$\lim_{m_i \to 0} R\left(\frac{m_H^2}{Q^2}, \frac{m_g^2}{Q^2}\right) \to 0. \tag{5.7.14}$$

Subject to these constraints on R, the convergence of the series for $\widetilde{S}_g$ and $\widetilde{S}_H$, now calculated using

$$\widetilde{S}_H = \widehat{V}_H(\Gamma^{-1})_{HH} + \widehat{V}_g(\Gamma^{-1})_{gH} \tag{5.7.15}$$

$$\widetilde{S}_g = \widehat{V}_H(\Gamma^{-1})_{Hg} + \widehat{V}_g(\Gamma^{-1})_{gg}, \tag{5.7.16}$$

is not affected by R (within reason, of course). Also recall that to all orders F does not depend on $\widehat{S}_H$, so it does not depend on R to all orders. Thus if we are working to $O(a_s^p)$, a change in R (provided it satisfies equation (5.7.14)) will produce a change in F of $O(a_s^{p+1})$.

To ease the calculations of $\widehat{S}_H$ from diagrams, one can set R to zero. We will refer to this choice as the *Simplified VFNS*, or *S-VFNS*. The S-VFNS has been discussed to NLO in [85], where it is referred to as the S-ACOT scheme. Note that A_{00} is just $\widehat{S}_H$ when H is massless and m_H is simply a regulator of mass singularities.

Note that in the case that factorization is done in the $n_f = 0$ CWZ subscheme, which should be done when $Q^2 < O(m_H^2)$, equation (5.7.3) reads

$$F = \widetilde{S}_g f_g, \tag{5.7.17}$$

where

$$\widetilde{S}_g = \widehat{S}_g \Gamma_{gg}^{-1} \qquad f_g = \Gamma_{gg} f_{0g}, \tag{5.7.18}$$

so the freedom in the choice of $\widehat{S}_H$ does not allow us to make any simplifications in this case.

We now extend the above to the case of a gluon, $a(=3)$ light quarks and $6-a$ heavy quarks. Then, setting the bare heavy quark PDF's to zero, the cross section is given by

$$F = \sum_{i=0}^{a} \widehat{S}_i f_{0i}. \tag{5.7.19}$$

Now $O(m_{n_f}^2) < Q^2 < O(m_{n_f+1}^2)$. (Note we must always have $n_f \geq a$). Then we must factorize in the n_fth CWZ subscheme, where

$$F = \sum_{i=0}^{n_f} \widetilde{S}_i^{[n_f]} f_i^{[n_f]}, \tag{5.7.20}$$

where

$$f_{AP}^{[n_f]} = \sum_{j=0}^{a} \Gamma_{AP\ j}^{[n_f]} f_{0j} \tag{5.7.21}$$

$$f_{NPQ}^{[n_f]} = 0 \tag{5.7.22}$$

$$\widetilde{S}_{AP}^{[n_f]} = \sum_{j=0}^{n_f} \widehat{S}_j (\Gamma^{[n_f]-1})_{j\ AP}. \tag{5.7.23}$$

Note at this point that we do not need to calculate the $\widehat{S}_{NPQ}$ *at all*, even for factorization. Again, to all orders, F is invariant under changes in $\widehat{S}_K$ for $K = a+1, ..., 6$. This can be seen explicitly, as follows: Making the replacement

$$\widehat{S}_K \to \widehat{S}_K + \delta\widehat{S}_K, \tag{5.7.24}$$

and then substituting equations (5.7.21) and (5.7.23) into equation (5.7.20) we find the change in F to be

$$\begin{aligned} \delta F &= \sum_{j=a+1}^{n_f} \sum_{i=0}^{n_f} \sum_{k=0}^{a} \delta\widehat{S}_j \left(\Gamma^{[n_f]-1}\right)_{ji} \Gamma_{ik}^{[n_f]} f_{0k}^{[n_f]} \\ &= \sum_{j=a+1}^{n_f} \sum_{k=0}^{a} \delta\widehat{S}_j \delta_{jk} f_{0k}^{[n_f]} = 0 \end{aligned} \tag{5.7.25}$$

as we set out to show. Thus we are free to replace $\widehat{S}_K$ for $K = a+1, ..., 6$ with any other set of functions. However, we wish the perturbation series for all the $\widetilde{S}_i$ in equation (5.7.23) to be convergent, so, letting the functions that will replace the $\widehat{S}_K$ for $K = a+1, ..., n_f$ be called $\widehat{V}_K$, the AL's (i.e. terms of $O\left(\ln^n \frac{Q^2}{m_{AP}^2}\right)$) in $\sum_{I=a+1}^{n_f} \widehat{S}_I (\Gamma^{[n_f]-1})_{IK}$ must be the same as those in $\sum_{I=a+1}^{n_f} \widehat{V}_I^{n_f} (\Gamma^{[n_f]-1})_{IK}$.

Now, since $O(m_{n_f}^2) < Q^2 < O(m_{n_f+1}^2)$, again setting $\mu_R^2 = k_R Q^2$, we must expand $\widehat{S}_K$ for $K = a+1, ..., n_f$ in the form

$$\widehat{S}_K = \left(\sum_{j_0=0}^{\infty} \cdots \sum_{j_6=0}^{\infty}\right) \left(\prod_{i=0}^{n_f} \left(\frac{m_i^2}{Q^2}\right)^{j_i}\right) \left(\prod_{l=n_f+1}^{6} \left(\frac{Q^2}{m_l^2}\right)^{j_l}\right)$$
$$\times A_{j_0 \ldots j_6}^{K,n_f} \left(\left\{\ln \frac{Q^2}{m_{AP}^2}\right\}, \left\{\ln \frac{Q^2}{m_{NPQ}^2}\right\}\right), \tag{5.7.26}$$

Note $A_{0000000}^{K,n_f}$ contains only AL's since the n_fth CWZ renormalization subscheme has been used, i.e. there are no terms of $O\left(\ln^j \frac{Q^2}{m_{NPQ}^2}\right)$ in $A_{0000000}^{K,n_f}$. Apart from the $A_{0000000}^{K,n_f}$ term, all mass logarithms in $\widehat{S}_K$ appear in the form $\left(\frac{m_{AP}^2}{Q^2}\right)^n \ln^m \frac{Q^2}{m_{AP}^2}$ for $n \geq 1$, and in the form $\left(\frac{Q^2}{m_{NPQ}^2}\right)^n \ln^m \frac{Q^2}{m_{NPQ}^2}$ (where $n \geq 1$), and so are not large. Incidentally, note this places some constraints on the logarithms contained in the $A_{j_0 \ldots j_6}^{K,n_f}$. Thus, in the limits $m_{AP} \to 0$ and $m_{NPQ} \to \infty$,

$$\widehat{S}_K = A_{0000000}^{K,n_f} + O\left(\left(\frac{m_{AP}^2}{Q^2}\right)^2\right) + O\left(\left(\frac{Q^2}{m_{NPQ}}\right)^2\right). \tag{5.7.27}$$

Thus all the large mass logarithms of the $\widehat{S}_K$ required in equation (5.7.23) are contained in $A_{0000000}^{K,n_f}$. Note that $A_{0000000}^{K,n_f}$ is just $\widehat{S}_K$ calculated in a theory with just n_f massless quarks and a gluon, where these partons are each given a small mass to regulate collinear divergences. Thus we are free to set

$$\widehat{V}_K^{n_f} = A_{0000000}^{K,n_f} + R_K^{n_f}, \; K = a+1, ..., n_f \tag{5.7.28}$$

where $R_K^{n_f}$ are any functions of our choosing but which must obey

$$\lim_{m_{AP} \to 0} R_K^{n_f} \left(\left\{\frac{m_{AP}^2}{Q^2}\right\}, \left\{\frac{Q^2}{m_{NPQ}^2}\right\}\right) \to 0 \tag{5.7.29}$$

$$\lim_{m_{NPQ} \to \infty} R_K^{n_f} \left(\left\{\frac{m_{AP}^2}{Q^2}\right\}, \left\{\frac{Q^2}{m_{NPQ}^2}\right\}\right) \to 0. \tag{5.7.30}$$

This means we can set $R_K^{n_f} = 0$ for all K, and we shall call this particular choice the S-VFNS.

All of the above also applies to $\widehat{F}_K$, so that we may set the X_i of equation (5.1.11) to unity.

Note that F calculated in the S-VFNS and F calculated in the VFNS are identical to all orders, assuming we defined F to be given by equation (5.7.19). Furthermore note that the factorization and renormalization schemes are the same in both cases. Thus everything else discussed in this chapter, e.g. matching conditions on and evolution of the f_i, are still the same, since $\Gamma^{[n_f]}$ and $a_s^{\{n_f\}}$ are the same in both schemes. The difference between the S-VFNS and the VFNS is merely the freedom in the choice of $R_K^{n_f}$ in equation (5.7.28), not the factorization and renormalization schemes. Since the S-VFNS and the VFNS are equal to all orders, and assuming they are both convergent, then when working to $O(a_s^p)$ in both schemes they differ by $O(a_s^{p+1})$. Finally, from equations (5.7.28), (5.7.29) and (5.7.30), the S-VFNS and the VFNS are the same when all AP masses go to zero and all NPQ masses go to infinity, so, like the VFNS, the S-VFNS also reduces to the ZM-VFNS in these limits.

The part of the leading region of $\widehat{F}$ we have not included is $\sum_{I=a+1}^{6} \widehat{S}_I f_{0I}$, since it is of $O\left(\frac{\Lambda_{QCD}^2}{m_{a+1}^2}\right)$ ($m_{a+1} = m_c$). If one decided that this part was relevant (which could be the case since it can be greater than r when $Q > O(m_{a+1})$), one could simply add it to the S-VFNS $\widehat{F}$. In this case F depends on the $\widehat{S}_K$ for $K = a+1, ..., n_f$, and so these cannot be replaced by simpler functions. However, they are not needed to as high an order in a_s as the $\widehat{S}_i$ for $i = 0, ..., a$ are, since the $\widehat{S}_K$ for $K = a+1, ..., n_f$ need only be calculated for the $\sum_{I=a+1}^{6} \widehat{S}_I f_{0I}$ part of F, which is of $O\left(\frac{\Lambda_{QCD}^2}{m_{a+1}^2}\right) \sim a_s(M_Z^2)$. Note the S-VFNS can only be used for light hadrons, since only then is equation (5.7.1) obeyed. For heavy hadrons (i.e. hadrons which are not light hadrons), we cannot make the simplifications described above.

5.8 Calculations To NLO

$\widehat{S}_{2I}^{(0)}$ has been calculated in [45] from the diagram in figure 2.3, and is given by

$$\widehat{S}_{2I}^{(0)}\left(\eta', Q^2\right) = \eta' e_{q_I}^2 \delta\left(1-\eta'\chi_I\right)\sqrt{1+\frac{4m_I^2}{Q^2}}. \tag{5.8.1}$$

χ_I is obtained by setting $\hat{s}_{th} = m_I^2$ in equation (5.1.7), to give

$$\chi_I = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4m_I^2}{Q^2}}\right). \tag{5.8.2}$$

Note as before that there is no gluon contribution, i.e.

$$\widehat{S}_{2g}^{(0)}\left(\eta', Q^2\right) = 0. \tag{5.8.3}$$

Note that after factorization, $\widetilde{S}_{2i}^{(0)}$ is scheme independent, so

$$\widetilde{S}_{2i}^{(0)} = \widehat{S}_{2i}^{(0)} \tag{5.8.4}$$

$$\widetilde{S}_{2g}^{[n_f](0)} = 0. \tag{5.8.5}$$

Thus to LO we find

$$F_2(\eta, Q^2) = \theta\left(1-\chi_I\eta\right)\eta\Omega(\eta, Q^2)\sum_{I=1}^{6}\sqrt{1+\frac{4m_I^2}{Q^2}}e_{q_I}^2 f_I^+(\eta\chi_I). \tag{5.8.6}$$

For the intrinsic light part of F_2 ($\sum_{i=1}^{a}\widehat{S}_i f_{0i}$) for light hadron scattering, using the results of section 5.7, we can replace $\widehat{S}_{2I}^{(0)}$ with its value when $m_I^2 = 0$, and so from equation (5.8.4),

$$\widetilde{S}_{2I}^{[n_f](0)} = e_I^2. \tag{5.8.7}$$

$\widehat{S}_{2I}^{(1)}$ has been obtained in [86]. However, in the intrinsic light part of F_2, we can use the results of section 5.7 to set $m_I^2 = 0$ everywhere in $\widehat{S}_{2I}^{(1)}$ except for the AL's (without needing the explicit expression for $\widehat{S}_{2I}^{(1)}$), and then, using equation (5.5.14) with the result for the n_fth CWZ subscheme that, as found in [86, 87],

$$\Gamma_{JI}^{[n_f](1)}(\eta', \mu_F^2) = \delta_{JI}\theta(n_f - I)\Bigg(P_{qq}^{(0)}(\eta') \ln \frac{\mu_F^2}{m_I^2} - C_F \left[\frac{1 + (\eta')^2}{1 - \eta'}(1 + 2\ln(1 - \eta'))\right]_+\Bigg), \tag{5.8.8}$$

where

$$\theta(n_f - I) = \begin{cases} 1 & \text{for } I \leq n_f \\ 0 & \text{for } I > n_f, \end{cases} \tag{5.8.9}$$

the value for $\widetilde{S}_{2I}^{[n_f](1)}$ that we would obtain must be that for n_f massless flavours in the $\overline{MS}$ scheme,

$$\widetilde{S}_{2I}^{[n_f](1)}\left(\eta', Q^2\right) = \theta(n_f - I)e_{qI}^2 \eta' C_F \left[\frac{1 + \eta'^2}{1 - \eta'}\left(\ln \frac{1 - \eta'}{\eta'} - \frac{3}{4}\right) + \frac{1}{4}(9 + 5\eta')\right]_+. \tag{5.8.10}$$

We now calculate $\widetilde{S}_{2g}^{[n_f](1)}$. $\widehat{S}_{2g}^{(1)}$ is obtained from the diagram in figure 2.6, and 3 other similar diagrams. From [76],

$$\widehat{S}_{2g}^{(1)}\left(\eta', Q^2\right) = \sum_{I=1}^{6} \eta' e_{qI}^2 \theta(1 - \chi_{g,I}\eta')$$
$$\Bigg[\left(\ln \frac{Q^2}{m_I^2} + \ln\left[\frac{1}{2}\left(1 - \eta' + \sqrt{(1 - \eta')\left(1 - \eta' - 4\eta' \frac{m_I^2}{Q^2}\right)}\right) - \eta' \frac{m_I^2}{Q^2}\right] - \ln \eta'\right)$$
$$\times \left(\eta'^2 + (1 - \eta')^2 + 4\frac{m_I^2}{Q^2}\left(\eta' - 3\eta'^2 - 2\eta'^2 \frac{m_I^2}{Q^2}\right)\right)$$
$$+ \sqrt{(1 - \eta')\left(1 - \eta' - 4\eta' \frac{m_I^2}{Q^2}\right)}\left(4\eta'\left(1 - \frac{m_I^2}{Q^2}\right) - \frac{(1 - 2\eta')^2}{1 - \eta'}\right)\Bigg]. \tag{5.8.11}$$

$\chi_{g,I}$ is found by setting $\hat{s}_{\mathrm{thg},I} = 4m_I^2$ and $m_g = 0$ in equation (5.1.7), to give

$$\chi_{g,I} = 1 + \frac{4m_I^2}{Q^2}. \tag{5.8.12}$$

Using equation (5.5.15), and using the simpler form

$$\sum_{i=0}^{6} \int_{\eta}^{1} d\xi A_i(\eta') B_i(\xi) = (A \otimes B)(\eta) \tag{5.8.13}$$

to write our equations, we will see just now that the correct subtraction term in the n_fth CWZ factorization subscheme, in η space, is

$$\left(\widetilde{S}_2^{(0)} \otimes \Gamma^{[n_f](1)}\right)_g \left(\eta', Q^2, \mu_F^2\right) = \sum_{I=1}^{n_f} e_I^2 \left[2P_{qg}^{(0)}(\eta') \ln \frac{\mu_F^2}{m_I^2}\right], \tag{5.8.14}$$

where

$$P_{qg}^{(0)}(x) = T_R[(1-x)^2 + x^2]. \tag{5.8.15}$$

In N space this becomes

$$\left(\widetilde{S}_2^{(0)}(N-1, Q^2)\Gamma^{[n_f](1)}(N, \mu_F^2)\right)_g = \sum_{I=1}^{n_f} e_I^2 \left[2P_{qg}^{(0)}(N) \ln \frac{\mu_F^2}{m_I^2}\right]. \tag{5.8.16}$$

We have therefore taken

$$\Gamma_{Ig}^{[n_f](1)}(N, \mu_F^2) = 2\theta(n_f - I) P_{qg}^{(0)}(N) \ln \frac{\mu_F^2}{m_I^2}. \tag{5.8.17}$$

To verify equation (5.8.17) we carry out the subtraction in η space to give

$$\widetilde{S}_{2g}^{[n_f](1)}\left(\eta', Q^2\right) = \sum_{I=1}^{6} \eta' e_{q_I}^2 \theta(1 - \chi_{g,I}\eta')$$

$$\left[\left(\ln \frac{Q^2}{m_I^2} + \ln\left[\frac{1}{2}\left(1 - \eta' + \sqrt{(1-\eta')\left(1 - \eta' - 4\eta' \frac{m_I^2}{Q^2}\right)}\right) - \eta' \frac{m_I^2}{Q^2}\right] - \ln \eta'\right)\right.$$

$$\times \left(\eta'^2 + (1-\eta')^2 + 4\frac{m_I^2}{Q^2}\left(\eta' - 3\eta'^2 - 2\eta'^2\frac{m_I^2}{Q^2}\right)\right)$$
$$+\sqrt{(1-\eta')\left(1-\eta'-4\eta'\frac{m_I^2}{Q^2}\right)}\left(4\eta'\left(1-\frac{m_I^2}{Q^2}\right) - \frac{(1-2\eta')^2}{1-\eta'}\right)\Bigg]$$
$$-\sum_{I=1}^{n_f}\eta' e_{qI}^2\left[2P_{qg}^{(0)}(\eta')\ln\frac{\mu_F^2}{m_I^2}\right]. \tag{5.8.18}$$

As $\frac{m^2}{Q^2} \to 0$ for all AP masses, equation (5.8.18) is non singular. As $\frac{m^2}{Q^2} \to \infty$ for NPQ masses, the NPQ's decouple from equation (5.8.18), because the argument of the θ function in equation (5.8.18) is zero when $\chi_{g,I} > \frac{1}{\eta'}$. Note at this point that we can see that at small η', a heavy quark I does not decouple very easily even if $m_I^2 \gg Q^2$. For both these limits on all AP and NPQ masses, equation (5.8.18) becomes

$$\tilde{S}_{2g}^{[n_f](1)}(\eta') = \sum_{I=1}^{n_f} e_{qI}^2\eta'\left[\ln\left(\frac{1}{\eta'}-1\right)\left(2\eta'^2-2\eta'+1\right)+8\eta'-8\eta'^2-1\right]. \tag{5.8.19}$$

Thus from [34] we can write

$$\tilde{S}_{2g}^{[n_f](1)}(\eta') = \sum_{I=1}^{n_f} e_{qI}^2\eta' C_g^{\overline{MS}(1)}(\eta'). \tag{5.8.20}$$

Thus equation (5.8.17) is the correct form for $\Gamma_{Ig}^{[n_f](1)}$ in the n_fth CWZ factorization scheme.

Equations (5.8.18) and (5.8.20), divided by η', are plotted in figures 5.1 to 5.4 for different values of Q^2 and n_f. The label "0" refers to the ZM-VFNS, and the label "1" refers to the VFNS. Note that mass effects tend to reduce the small η' logarithmic singularity. However, the VFNS gluon coefficient function contains a $\frac{1}{1-\eta'}$ singularity as $\eta' \to 1$, not present in the ZM-VFNS case. Furthermore, note that the VFNS gluon coefficient function contains discontinuities due to the step function in equation (5.8.18). The ZM-VFNS gluon coefficient function is of course independent of Q^2, but not n_f.

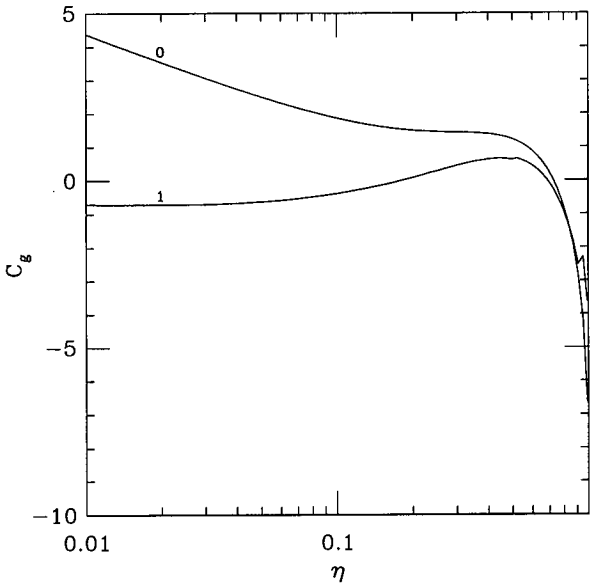

Figure 5.1: Gluon coefficient functions for $n_f = 5$ and $Q^2 = 100$.

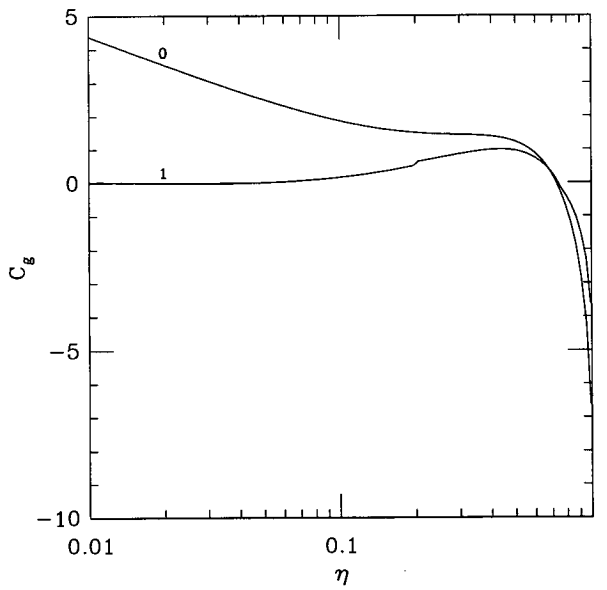

Figure 5.2: Gluon coefficient functions for $n_f = 5$ and $Q^2 = 25$.

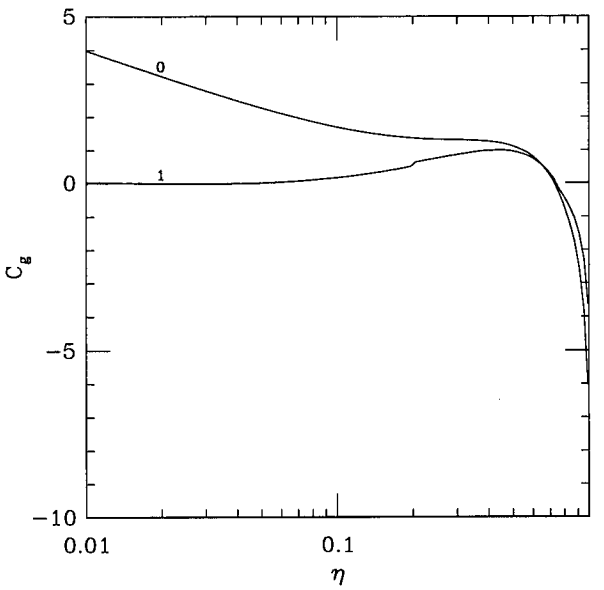

Figure 5.3: Gluon coefficient functions for $n_f = 4$ and $Q^2 = 25$.

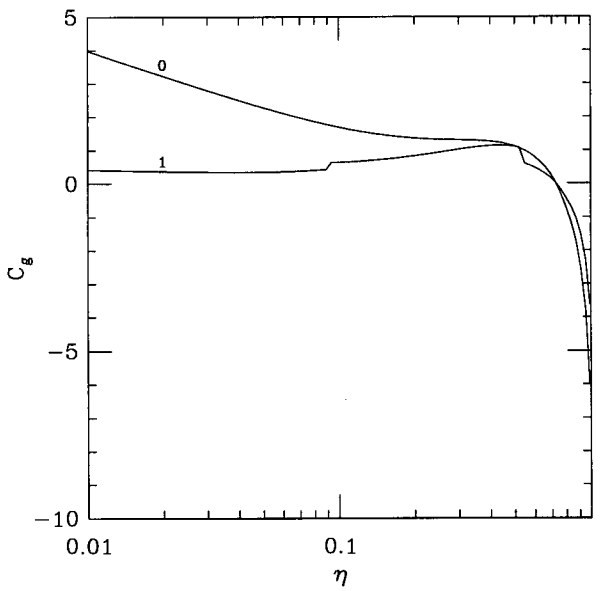

Figure 5.4: Gluon coefficient functions for $n_f = 4$ and $Q^2 = 10$.

5.9 Factorization And Renormalization Scale Dependence

In the CWZ factorization and renormalization subschemes, the results of section 2.18 apply in the same way, since the evolution kernel E and a_s are the same. However, the matching conditions at threshold need careful treatment. We now impose equations (2.18.2) and (2.18.3). To ensure that the number of flavours used in both factorization and renormalization is the same, which we would expect to be the most physical case, we impose equation (2.19.8). We first write $A^{[n_f]}$ to NNLO in the form

$$A^{[n_f]\{n_f\}}(N,\mu_F^2) = \mathbb{1}+a_s^{\{n_f\}}(\mu_F^2)A^{[n_f]\{n_f\}(1)}(N,\mu_F^2)+a_s^{\{n_f\}2}(\mu_F^2)A^{[n_f]\{n_f\}(2)}(N,\mu_F^2). \tag{5.9.1}$$

From equation (2.18.10),

$$a_s^{\{n_f\}}(m_i''^2) = a_s^{\{n_f\}}(m_i'^2) + a_s^{\{n_f\}2}(m_i'^2)\beta_0^{\{n_f\}} \ln\frac{k_R}{k_F}, \tag{5.9.2}$$

so that equation (5.9.1) reads

$$A^{[n_f]}(N,m_i''^2) = \mathbb{1} + a_s^{\{n_f\}}(m_i'^2)A^{[n_f]\{n_f\}(1)}\left(N,\frac{k_F}{k_R}m_i'^2\right)$$
$$+a_s^{\{n_f\}2}(m_i'^2)\left[A^{[n_f]\{n_f\}(2)}\left(N,\frac{k_F}{k_R}m_i'^2\right) + \beta_0^{\{n_f\}}\ln\frac{k_R}{k_F}A^{[n_f]\{n_f\}(1)}\left(N,\frac{k_F}{k_R}m_i'^2\right)\right]. \tag{5.9.3}$$

For the case that $\frac{k_F}{k_R}m_b'^2 > \mu_{F0}^2 > \frac{k_F}{k_R}m_c'^2$, we do the same as in section 2.19, with some small modifications. We evolve the singlet and gluon down from μ_{F0}^2 to $\frac{k_F m_c'^2}{k_R}$ as in equation (2.19.12), with $n_f = 4$. Then, from equations (5.6.16) and (5.6.18), and using the result from equation (5.6.14) that

$$f_g^{[n_f]}(N,m_{n_f}''^2) = f_g^{[n_f-1]}(N,m_{n_f}''^2) + O(a_s^2), \tag{5.9.4}$$

we find

$$T^{[4]}_{4^2-1}\left(N,\frac{k_F m_c'^2}{k_R}\right)=\Sigma^{[4]}\left(N,\frac{k_F m_c'^2}{k_R}\right)$$
$$-4a_s^{\{4\}}(m_c'^2)2P_{qg}^{(0)}(N)\ln\frac{k_F m_c'^2}{k_R m_c^2}f_g^{[4]}\left(N,\frac{k_F m_c'^2}{k_R}\right). \tag{5.9.5}$$

5.10 Converting To Mellin Space

The Mellin transform of equation (5.8.11) is analytically unknown, and numerically divergent for Re(N) below a certain value. However, we need to know our results for any N so that one can do the inverse Mellin transform, given by

$$F_2(\eta,Q^2)=\eta\Omega(\eta,Q^2)\frac{1}{2\pi i}\int_C dN\eta^{-N}F_2^{Norm}(N-1,Q^2), \tag{5.10.1}$$

where the contour C was discussed in section 2.2. A way around these problems is to fit the part of equation (5.8.11) for which we do not know the Mellin transform to polynomials in η' space, and then analytically Mellin transform the polynomials. However, our results contain discontinuities (step functions and delta functions) so that some analytic work is needed before we can think about the fitting. Firstly, equation (5.8.11) is of the form

$$S(\eta')=S'(\eta')\,\theta(1-\chi\eta'), \tag{5.10.2}$$

so that its Mellin transform is given by

$$S(N)=\chi^{-N}\int_0^1 dy y^{N-1}S'\left(\frac{y}{\chi}\right). \tag{5.10.3}$$

However, the function S' above cannot be fitted to polynomials, since $S'(z)$ is not finite for the region we wish to fit to, $0\leq z\leq\frac{1}{\chi}$. We can get around this problem as follows: Firstly, the S' in equation (5.10.2) can be written, by close inspection of equation (5.8.11), in the form

$$S'(z) = S''(z) + \ln z S'''(z), \tag{5.10.4}$$

where $S''(z)$ and $S'''(z)$ are finite over the required range $0 \leq z \leq \frac{1}{\chi}$. Then equation (5.10.3) becomes

$$S(N) = \chi^{-N} \int_0^1 dy y^{N-1} \Big[S'' \left(\frac{y}{\chi}\right) + \ln \frac{1}{\chi} S''' \left(\frac{y}{\chi}\right) + \ln y S''' \left(\frac{y}{\chi}\right) \Big]. \tag{5.10.5}$$

We can find the Mellin transform of $S''' \left(\frac{y}{\chi}\right)$ and $\ln y S''' \left(\frac{y}{\chi}\right)$ analytically, and one can fit polynomials to the function $S'' \left(\frac{y}{\chi}\right)$ of the first line, as this function is finite over the range $0 \leq y \leq 1$. Then one can take the Mellin transform of the polynomials using the result

$$y^n \to \frac{1}{N+n}. \tag{5.10.6}$$

We will use Chebyshev polynomials for the fits to $S'' \left(\frac{y}{\chi}\right)$, which are given by

$$T_0(x) = 1 \qquad T_1(x) = x \tag{5.10.7}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{5.10.8}$$

The $T_n(x)$ are defined in the range $-1 \leq x \leq 1$. A function $f(x)$ is approximated in the following way: Defining coefficients c_j by

$$c_j = \frac{2}{M} \sum_{k=1}^{M} f\left(\cos\left(\frac{\pi(k-\frac{1}{2})}{M}\right)\right) \cos\left(\frac{\pi(j-1)(k-\frac{1}{2})}{M}\right), \tag{5.10.9}$$

we obtain the approximation

$$f(x) \simeq \left[\sum_{k=1}^{M} c_k T_{k-1}(x)\right] - \frac{c_1}{2}. \tag{5.10.10}$$

The condition on $f(x)$ is that it must remain finite in the range $-1 \leq x \leq 1$. For the functions we wish to fit polynomials to, we map their arguments y onto x via the equation

$$y = \frac{1}{2}(x+1), \tag{5.10.11}$$

so that

$$0 \leq y \leq 1. \tag{5.10.12}$$

If $T'_n(N)$ is the Mellin transform with respect to y of the product of $T_n(x)$ with the relevant function of y, i.e. those functions that are the coefficients of y^n in equation (5.10.6), the recurrence relation in equation (5.10.8) becomes

$$T'_{n+1}(N) = 4T'_n(N+1) - 2T'_n(N) - T'_{n-1}(N). \tag{5.10.13}$$

When $n \gtrsim 20$, machine precision means this method is not very accurate if $y \gtrsim 0.1$. However, for the region we will fit PDF parameters to, the relative accuracy of the polynomial fit is around 0.005 for $n = 20$, which is similar to the relative accuracy due to truncation, since $a_s(M_Z^2) \sim 0.01$.

Finally, before we can perform the inverse Mellin transform as specified in equation (5.10.1), there is one more thing we need to take care of. For a given process, or Feynman diagram, the contribution to F_2^{Norm} can be generically written

$$F(\eta) = \int_{\eta}^{1} d\xi S(\eta') f(\xi), \tag{5.10.14}$$

where $S(\eta')$ can be written

$$S(\eta') = \theta(1 - \chi\eta')\, S'(\eta'). \tag{5.10.15}$$

Equation (5.10.14) vanishes when $\eta\chi > 1$. Thus we need to check that if we take equation (5.10.14) to Mellin space, as we will, and then transform back to η space, we do indeed get $F(\eta) = 0$ when $\eta\chi > 1$. Now equation (5.10.14) in Mellin space becomes

$$F(N-1) = S(N-1)f(N). \tag{5.10.16}$$

From equation (5.10.15), $S(N-1)$ can be written

$$S(N-1) = \int_0^{\frac{1}{\chi}} d\eta' \eta'^{N-2} S(\eta'). \tag{5.10.17}$$

Then, to transform back to η space we must use

$$F(\eta) = \eta \frac{1}{2\pi i} \int_C dN \eta^{-N} S(N-1) f(N). \tag{5.10.18}$$

Using equation (5.10.17), we can rewrite equation (5.10.18) as

$$F(\eta) = \eta \frac{1}{2\pi i} \int_C dN (\chi\eta)^{-N} \chi \int_0^1 dy y^{N-2} S\left(\frac{y}{\chi}\right) f(N). \tag{5.10.19}$$

Now as $\mathrm{Re}(N) \to -\infty$ along the contour C, this integration converges if $\chi\eta < 1$ due to the $(\chi\eta)^{-N}$ term. However, if $\chi\eta > 1$ the integration diverges, but we know that $F(\eta)$ is zero for $\chi\eta > 1$, not divergent. ($f(N)$ and the integration over $S\left(\frac{y}{\chi}\right)$ are only of $O\left(\frac{1}{N}\right)$ for large negative N). Thus, when $\chi\eta > 1$, we must set $F(\eta)$ to zero by hand in calculations.

5.11 Fitting To Experimental Data

As mentioned at the end of section (5.7), the LO contribution of the factorized intrinsic heavy quark PDF's can be of the same order of magnitude as the $O(a_s)$ part of the cross section. However, the usual ZM-VFNS already give good fits to large and intermediate values of η without needing more theoretical input, and our approximation to the gluon coefficient function is unstable at small η. Our interest here will be in the effect of varying the threshold scale when fitting PDF's and higher twist to BCDMS and SLAC data in the VFNS compared with the ZM-VFNS.

When Q (or μ_R or μ_F) is of the order of a quark mass, in the VFNS (or S-VFNS) to infinite order it does not make any difference whether the quark is chosen to be active or non-partonic, since the result is always the same. However, to infinite order in the ZM-VFNS, the result clearly *does* depend on whether the quark has been decoupled. Thus, one would expect greater threshold scale dependence in the ZM-VFNS than in the VFNS to NLO.

We shall take

$$m_i'' = m_i' \tag{5.11.1}$$

for all i, and, defining

$$t_s = \frac{m_i'}{m_i}, \tag{5.11.2}$$

we perform fits of PDF's and higher twist for $t_s = 1, 4$ and $\frac{1}{4}$. We consider $t_s = 1$ and $\frac{1}{4}$ to examine the threshold scale dependence of fitted PDF's, since for $t_s = 1$ and $\frac{1}{4}$ the PDF's are in the 4th CWZ factorization subscheme in both cases, whereas for $t_s = 4$ the PDF's are in the 3rd CWZ factorization subscheme. We consider $t_s = 4$ to examine higher twist (which is scheme independent), since the case $t_s = \frac{1}{4}$ gives a large $a_s(m_c') \sim 1.5$ and therefore large theoretical errors. The constraints and conditions on PDF's, $\alpha_s(M_Z^2)$ and data fitted to are the same as those in section 4.6. For the pole masses, we took $m_c = 1.5$ GeV and $m_b = 5$

GeV. The light parton pole masses were set to zero. Since their masses can be as high as $O(\Lambda_{QCD})$, this means that our higher twist fits contain some leading order contributions. However, it will only be our purpose to consider relative differences between the VFNS and the ZM-VFNS. The results for the fitted parameters are shown in tables E.1.1 to E.1.6. Note that the error matrix was not always positive definite. The corresponding PDF plots are shown in figures E.2.1 to E.2.8. The label "5" implies $t_s = \frac{1}{4}$ was taken, "1" implies $t_s = 1$ and "2" implies $t_s = 4$. The fitted higher twists are shown in figures E.3.1 to E.3.8.

Figures E.4.1 to E.4.6 show the F_2 cross sections at different η as a function of Q^2 with different $t_s = 1, 4, \frac{1}{4}$ in the VFNS and the ZM-VFNS. We use the PDF's fitted in the VFNS with $t_s = 1$ for all these plots.

5.12 Conclusions

The discontinuity in the F_2 cross section when crossing a threshold in the VFNS is much less than in the ZM-VFNS. This is because in the ZM-VFNS the value of n_f in the coefficient function changes, whereas in the VFNS the coefficient functions are quite similar. F_2 in the VFNS is also much less sensitive to variations in t_s than F_2 in the ZM-VFNS, which may be helped by the discontinuous matching conditions of the PDF's at threshold. However, this sensitivity in the ZM-VFNS is really only large for $\eta < 0.1$.

The higher twist plots for $t_s = \frac{1}{4}$ show quite different behaviour to the plots for $t_s = 1, 4$, which we attribute to large theoretical errors in the former case. For the latter case, for $\eta < 0.3$, the VFNS gives a less negative higher twist than the ZM-VFNS. We infer that in the ZM-VFNS, the higher twist attempts to compensate for the large gluon coefficient function at small ξ. For $\eta \sim 1$, the VFNS appears to reduce the higher twist, possibly due to a singularity in the VFNS gluon coefficient function of the form $\frac{1}{1-\eta'}$ not present in the ZM-VFNS case.

The gluon PDF at small ξ is only slightly larger in the VFNS, despite the smaller gluon coefficient function. This may be partly a result of imposing the momentum

sum rule, given that the gluon momentum fraction can be so large in the VFNS.

The slightly larger gluon PDF and the momentum sum rule in the VFNS case helps to give a lower singlet in the VFNS than in the ZM-VFNS. However, the other quark PDF's appear to be lower in the VFNS. We suspect this is mainly caused by the following: For $t_s = 1, \frac{1}{4}$, charm is "removed" from the gluon coefficient function and absorbed into the initial PDF's. In the VFNS, this charm may then have the effect of pushing the other flavour PDF's down more so than in the ZM-VFNS. Certainly for $t_s = 4$ the singlet and non singlet PDF's in the VFNS are more similar to those in the ZM-VFNS, and here charm is not included in the initial PDF's.

The VFNS does not make a lot of improvement over the ZM-VFNS when fitting to SLAC and BCDMS data. This is due mainly to the fact that the correct inclusion of heavy quarks at large η and high Q^2 does not change the F_2 cross section over the ZM-VFNS very much, and the SLAC and BCDMS have their largest errors at small η and small Q^2. Secondly, PDF and higher twist parameterization may be sufficient at $\eta > 0.1$ to compensate for heavy quark effects in SLAC and BCDMS data.

Appendix A

Statistics

A.1 Systematic Errors

Consider an experiment E consisting of N experimental measurements f_i^e, $i = 1, ..., N$, of true values f_i^t, with N_s sources of systematic errors. Each measurement f_i^e can be written as

$$f_i^e = f_i^t + \gamma_i \sigma_i^{\text{stat}} + \sum_{K=1}^{N_s} \lambda_K \sigma_i^K, \tag{A.1.1}$$

where the σ_i^{stat} and the σ_i^K are the statistical and systematic errors respectively. The γ_i and λ_K are the statistical and systematic parameters respectively, distributed according to an *a priori* probability density $P(\boldsymbol{\lambda}, \boldsymbol{\sigma})$. These parameters are statistically independent of each other, and normalized to variance 1. Hence their covariances are given by

$$\text{cov}(\gamma_i, \gamma_j) = \delta_{ij} \qquad \text{cov}(\lambda_K, \lambda_L) = \delta_{KL} \qquad \text{cov}(\gamma_i, \lambda_K) = 0, \tag{A.1.2}$$

where e.g.

$$\text{cov}(\gamma_i, \lambda_K) = \int d\boldsymbol{\lambda} d\boldsymbol{\sigma} P(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \gamma_i \lambda_K$$

$$- \left(\int d\boldsymbol{\lambda} d\boldsymbol{\sigma} P(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \gamma_i \right) \left(\int d\boldsymbol{\lambda} d\boldsymbol{\sigma} P(\boldsymbol{\lambda}, \boldsymbol{\sigma}) \lambda_K \right). \tag{A.1.3}$$

Note the λ_K are fixed but unknown throughout an experiment. The f_i^e are then correlated with each other, and this correlation can be described by the covariance matrix $C_{ij} = \mathrm{cov}(f_i^e, f_j^e)$, which, from equations (A.1.1) and (A.1.2), is given by

$$C_{ij} = \sigma_i^{\mathrm{stat}} \sigma_j^{\mathrm{stat}} \delta_{ij} + \sum_K \sigma_i^K \sigma_j^K. \tag{A.1.4}$$

A.2 Definition of χ^2

From now on, we will only consider the case where the γ_i and λ_K are normally distributed. Because of equation (A.1.1), each f_i^e is normally distributed with mean f_i^t. Therefore, because of the shared value of each λ_K between the f_i^e, the set of measurements $\mathbf{f}^e$ will have a probability density of a correlated N dimensional normal distribution given by

$$P(\mathbf{f}^e) = \frac{1}{\sqrt{(2\pi)^N |C|}} \exp\left[-\frac{1}{2} \chi^2 \right], \tag{A.2.1}$$

where

$$\chi^2 = \sum_{ij} (f_i^e - f_i^t)(C^{-1})_{ij} (f_j^e - f_j^t). \tag{A.2.2}$$

Equation (A.2.2) is the usual definition of χ^2 and its calculation, and this is the quantity we should minimize with respect to theoretical parameters. We will now show some other methods of obtaining theoretical parameters, whilst treating systematic errors, that the above procedure is equivalent to. This will also lead us to an expression for χ^2 that only involves the inversion of a matrix that is much smaller than C_{ij}. We will use the result from Bayes' Theorem that the probability of the systematic parameters lying between λ_K and $\lambda_K + d\lambda_K$ and the experimental measurements lying between f_i^e and $f_i^e + df_i^e$ is given by

$$P(\mathbf{f}^e, \boldsymbol{\lambda}) d\mathbf{f}^e d\boldsymbol{\lambda} = P(\boldsymbol{\lambda}|\mathbf{f}^e) d\boldsymbol{\lambda}\ P(\mathbf{f}^e) d\mathbf{f}^e = P(\mathbf{f}^e|\boldsymbol{\lambda}) d\mathbf{f}^e\ P(\boldsymbol{\lambda}) d\boldsymbol{\lambda}. \tag{A.2.3}$$

$P(\boldsymbol{\lambda}|\mathbf{f}^e)$ is the conditional probability density that, for given values of f_i^e, the systematic parameters are λ_K. $P(\mathbf{f}^e|\boldsymbol{\lambda})$ is the conditional probability density that, for given values of λ_K, the measurements will be f_i^e. Note that all these probability densities depend on the particular experiment, and so are also functions of the σ_i^{stat} and σ_i^K.

Since the f_i^e are normally distributed around $f_i^t + \sum_K \lambda_K \sigma_i^K$ with variance σ_i^{stat}, we have

$$P(\mathbf{f}^e|\boldsymbol{\lambda}) = \frac{1}{\sqrt{(2\pi)^N} \prod_j \sigma_j^{\text{stat}}} \exp\left[-\frac{1}{2} \sum_j \left(\frac{f_i^e - f_i^t - \sum_K \lambda_K \sigma_i^K}{\sigma_i^{\text{stat}}} \right)^2 \right]. \tag{A.2.4}$$

If the λ_K are normally distributed, we have

$$P(\boldsymbol{\lambda}) = \frac{1}{\sqrt{(2\pi)^{N_s}}} \exp\left[-\frac{1}{2} \sum_K \lambda_K^2\right]. \tag{A.2.5}$$

A.3 Integration over the λ_K

Integrating equation (A.2.3) over the λ_K gives

$$P(\mathbf{f}^e) = \int d\boldsymbol{\lambda} P(\mathbf{f}^e|\boldsymbol{\lambda}) P(\boldsymbol{\lambda}). \tag{A.3.1}$$

Substituting equations (A.2.4) and (A.2.5) into equation (A.3.1), we find

$$P(\mathbf{f}^e) = \frac{1}{\sqrt{(2\pi)^{N+N_s}} \Pi_j \sigma_j^{\text{stat}}} \int d\boldsymbol{\lambda} \exp\left[-\frac{1}{2}\chi_\lambda^2\right], \tag{A.3.2}$$

where

$$\chi^2_\lambda = \sum_i \left(\frac{f_i^e - f_i^t - \sum_K \lambda_K \sigma_i^K}{\sigma_i^{\mathrm{stat}}} \right)^2 + \sum_K \lambda_K^2. \tag{A.3.3}$$

Defining

$$A_i^K = \frac{\sigma_i^K}{\sigma_i^{\mathrm{stat}}} \qquad\qquad B_i = \frac{f_i^e - f_i^t}{\sigma_i^{\mathrm{stat}}}, \tag{A.3.4}$$

equation (A.3.3) can be more simply written as

$$\chi^2_\lambda = (B_i - \lambda_K A_i^K)(B_i - \lambda_L A_i^L) + \lambda_K \lambda_K, \tag{A.3.5}$$

where, in equation (A.3.5) and from now on, repeated indices are to be summed over. Thus

$$\chi^2_\lambda = B_i B_i - 2\lambda_K A_i^K B_i + \lambda_K H_{KL} \lambda_L, \tag{A.3.6}$$

where

$$H_{KL} = \delta_{KL} + A_i^K A_i^L. \tag{A.3.7}$$

We define new quantities α_L via the equation

$$\lambda_K = R_{KL} \alpha_L, \tag{A.3.8}$$

where R_{KL} is such that the $\lambda_K H_{KL} \lambda_L$ term in equation (A.3.6) becomes

$$\lambda_K H_{KL} \lambda_L = \alpha_K \alpha_K. \tag{A.3.9}$$

Substitution of equation (A.3.8) into equation (A.3.9) gives a condition on the matrix R:

$$R^T H R = I. \tag{A.3.10}$$

This means that

$$R R^T = H^{-1}. \tag{A.3.11}$$

Note that equation (A.3.11) does not determine R uniquely. Substitution of equation (A.3.8) into equation (A.3.6) yields

$$\chi^2_\lambda = B_i B_i - 2 R_{KL} \alpha_L A^K_i B_i + \alpha_K \alpha_K. \tag{A.3.12}$$

Completing the square, we find

$$\chi^2_\lambda = B_i B_i - B_i A^L_i R_{LK} B_j A^M_j R_{MK} + \left(\alpha_K - B_i A^L_i R_{LK}\right)\left(\alpha_K - B_j A^M_j R_{MK}\right), \tag{A.3.13}$$

and from equation (A.3.11),

$$\chi^2_\lambda = B_i B_i - B_i A^L_i (H^{-1})_{LM} B_j A^M_j + \left(\alpha_K - B_i A^L_i R_{LK}\right)\left(\alpha_K - B_j A^M_j R_{MK}\right). \tag{A.3.14}$$

Using equation (A.3.8) in the form $|d\boldsymbol{\lambda}/d\boldsymbol{\alpha}| = |R|$, equation (A.3.2) becomes

$$\begin{aligned} P(\mathbf{f}^e) &= \frac{1}{\sqrt{(2\pi)^{N+N_s} \Pi_j \sigma^{\mathbf{stat}}_j}} \int d\boldsymbol{\alpha} |R| \exp\left[-\frac{1}{2}\chi^2_\lambda\right] \\ &= \frac{1}{\sqrt{(2\pi)^{N+N_s} \Pi_j \sigma^{\mathbf{stat}}_j}} |R| \int d\boldsymbol{\alpha} \exp\left[-\frac{1}{2}\chi^2_\lambda\right] \end{aligned} \tag{A.3.15}$$

From equation (A.3.11), we find that $|R| = (\sqrt{|H|})^{-1}$, so that

$$P(\mathbf{f}^e) = \frac{1}{\sqrt{(2\pi)^{N+N_s} \Pi_j \sigma^{\mathbf{stat}}_j}\sqrt{|H|}} \int d\boldsymbol{\alpha} \exp\left[-\frac{1}{2}\chi^2_\lambda\right] \tag{A.3.16}$$

Performing the integral over $\boldsymbol{\alpha}$ using equation (A.3.14), we find

$$P(\mathbf{f}^e) = \frac{1}{\sqrt{(2\pi)^N \Pi_j \sigma_j^{\text{stat}} \sqrt{|H|}}} \exp\left[-\frac{1}{2}\chi^2\right], \tag{A.3.17}$$

where

$$\chi^2 = B_i B_i - B_i A_i^L (H^{-1})_{LM} B_j A_j^M. \tag{A.3.18}$$

In the usual case $N_s \ll N$, this form of χ^2 requires many less calculations to be evaluated than the form in equation (A.2.2), and only involves the inversion of the smaller matrix H_{KL}. That equation (A.3.18) yields the same result for χ^2 as equation (A.2.2) can be seen by rewriting equation (A.2.2) in the form

$$\chi^2 = B_i (Q^{-1})_{ij} B_j, \tag{A.3.19}$$

where $(Q^{-1})_{ij} = \sigma_i^{\text{stat}} \sigma_j^{\text{stat}} (C^{-1})_{ij}$. Therefore $Q_{ij} = C_{ij}/\sigma_i^{\text{stat}} \sigma_j^{\text{stat}}$, or from equation (A.1.4),

$$Q_{ij} = \delta_{ij} + A_i^K A_j^K \tag{A.3.20}$$

Comparing equations (A.3.18) and (A.3.19), we must have

$$(Q^{-1})_{ij} = \delta_{ij} - A_i^L (H^{-1})_{LM} A_j^M. \tag{A.3.21}$$

Equation (A.3.21) can be verified by matrix multiplication of equation (A.3.21) with Q, and then use of equation (A.3.20).

A.4 Minimization With Respect To The λ_K

Minimizing χ^2_λ with respect to the λ_K and the theoretical parameters is equivalent to just miminizing χ^2 with respect to the theoretical parameters, since the value

of each α_K that minimizes equation (A.3.13) is

$$\alpha_K = B_j A_j^M Q_{MK}, \tag{A.4.1}$$

and here $\chi^2_\lambda = \chi^2$. From equation (A.3.8), equation (A.4.1) implies

$$\lambda_K = Q_{KL} B_j A_j^M Q_{ML}. \tag{A.4.2}$$

A.5 Fitting Theoretical Parameters

In practice we do not know what the values of f_i^t are, so we need to find a model to predict them. For given f_i^t predicted from a model T, $P(\mathbf{f}^e)d\mathbf{f}^e$, which recall depends on the f_i^t, is then the probability of the experiment E obtaining the results f_i^e in the range $d\mathbf{f}^e$ given that a particular theoretical model T is true, which we write as $\Phi(\mathbf{f}^e|T)d\mathbf{f}^e$. Then from Bayes' Theorem, the probability of the theory being true given the experiment E with results f_i^e is

$$\Phi(T|\mathbf{f}^e)DT = \frac{P(\mathbf{f}^e)d\mathbf{f}^e \ \Phi(T)DT}{\Phi(\mathbf{f}^e)d\mathbf{f}^e}. \tag{A.5.1}$$

Here, $\Phi(T)DT$ is the probability of theory T lying in the range "DT". $\Phi(T)$ therefore has nothing to do with the experiment E, but can be interpreted as the result of fitting all possible models to previous experimental data. $\Phi(\mathbf{f}^e)d\mathbf{f}^e$ is the certainty of obtaining the f_i^e in the range $d\mathbf{f}^e$ in experiment E given our certainties in each and every theoretical model, and our knowledge of E. We need to define what we mean by DT. We suppose we know with complete confidence that a theory $T(\mathbf{a})$ is true, where apart from the N_a unspecified parameters a^α $T(\mathbf{a})$ is uniquely defined. In other words we are limiting the space of all theories to only those that are members of $T(\mathbf{a})$. Then we will take

$$DT = d\mathbf{a}, \tag{A.5.2}$$

which completes the definition of $\Phi(T(\mathbf{a}))$ and $\Phi(T(\mathbf{a})|\mathbf{f}^e)$. Suppose we have absolutely no phenomenological knowledge prior to our fitting the parameters a^α to the results of experiment E. Furthermore, we suppose that there is no reason why one infinitesimal region $d\mathbf{a}$ of our parameter space is any more or less probable than another other region of the same hypervolume. Then we can set

$$\Phi(T(\mathbf{a})) = \text{constant}. \tag{A.5.3}$$

Thus equation (A.5.1) becomes

$$\Phi(T(\mathbf{a})|\mathbf{f}^e) = P(\mathbf{f}^e). \tag{A.5.4}$$

Thus from equations (A.2.1) and (A.2.2),

$$\Phi(T(\mathbf{a})|\mathbf{f}^e) \propto \exp\left[-\frac{1}{2}\chi^2(\mathbf{a})\right], \tag{A.5.5}$$

where

$$\chi^2(\mathbf{a}) = \sum_{ij}(f_i^e - f_i^t(\mathbf{a}))(C^{-1})_{ij}(f_j^e - f_j^t(\mathbf{a})). \tag{A.5.6}$$

Let assume that the Taylor expansion

$$f_i^t(\mathbf{a}) = f_i^t(\mathbf{a}_0) + \sum_\alpha \left.\frac{\partial f_i^t}{\partial a^\alpha}\right|_{\mathbf{a}=\mathbf{a}_0} (a^\alpha - a_0^\alpha) + \frac{1}{2}\sum_{\alpha\beta} \left.\frac{\partial^2 f_i^t}{\partial a^\alpha \partial a^\beta}\right|_{\mathbf{a}=\mathbf{a}_0} (a^\alpha - a_0^\alpha)\left(a^\beta - a_0^\beta\right) \tag{A.5.7}$$

is a good approximation over the range of a^α we wish to investigate. We will define the a_0^α to be the values of the a^α for which $\chi^2(\mathbf{a})$ is minimized, i.e.

$$\left.\frac{\partial \chi^2}{\partial a^\alpha}\right|_{\mathbf{a}=\mathbf{a}_0} = 0. \tag{A.5.8}$$

Then from equations (A.5.6) and (A.5.7), the a_0^α are the solutions to the equations

$$\left(f_i^e - f_i^t(\mathbf{a}_0)\right)(C^{-1})_{ij}\frac{\partial f_j^t}{\partial a^\alpha}\Big|_{\mathbf{a}=\mathbf{a}_0} = 0. \tag{A.5.9}$$

Therefore

$$\chi^2 = \sum_{\alpha,\beta}(a^\alpha - a_0^\alpha)(C_a^{-1})_{\alpha\beta}(a^\beta - a_0^\beta) + \chi_0^2 + O(\{(a^\beta - a_0^\beta)\}^3), \tag{A.5.10}$$

where

$$(C_a^{-1})_{\alpha\beta} = \sum_{ij}\frac{\partial f_i^t}{\partial a^\alpha}\Big|_{\mathbf{a}=\mathbf{a}_0}(C^{-1})_{ij}\frac{\partial f_j^t}{\partial a^\beta}\Big|_{\mathbf{a}=\mathbf{a}_0} - \sum_{ij}\left(f_i^e - f_i^t(\mathbf{a}_0)\right)(C^{-1})_{ij}\frac{\partial^2 f_j^t}{\partial a^\alpha \partial a^\beta}\Big|_{\mathbf{a}=\mathbf{a}_0}. \tag{A.5.11}$$

and χ_0^2 is independent of the a^α. Note that the a_0^α are the values of the a^α which maximise the probability $\Phi(T(\mathbf{a})|\mathbf{f}^e)$, and therefore represent the most likely values of the a^α given the experiment E and results f_i^e. The values of the a_0^α can be calculated numerically by minimizing χ^2 as defined by equation (A.5.6). The values of the $C_{a,\alpha\beta}$ can be calculated numerically by varying each of the a^α around the a_0^α, assuming equation (A.5.10) to be a good approximation.

A.6 Reduced χ^2

To claim that our model is a suitable model for the data in question, from equation (A.1.1) we require

$$(f_i^e - f_i^t)^2 \lesssim \left(\sum_K \sigma_i^{K\ 2}\right) + \sigma_i^{\text{stat}\ 2} \quad \text{for all } i. \tag{A.6.12}$$

From equations (A.3.4) and (A.3.20), this means we require (noting that we are disposing of the summation convention in this section)

$$B_i \lesssim \sqrt{Q_{ii}}. \tag{A.6.13}$$

From equation (A.3.19), this means we require

$$\chi^2 \lesssim \sum_{i,j} \sqrt{Q_{ii}}(Q^{-1})_{ij}\sqrt{Q_{jj}}. \tag{A.6.14}$$

In fact equation (A.6.12) is rather a strong condition when correlated errors are large, since correlations can imply that an increase in $(f_i^e - f_i^t)^2$ for one value of i *must* give an increase in these quantities for all other i for the model to be good. If we assume the systematic errors are much smaller than the statistical ones, then $Q_{ij} \sim \delta_{ij}$, so that equation (A.6.14) implies

$$\chi^2 \lesssim N. \tag{A.6.15}$$

However, suppose we have n_t theoretical parameters in our model which we use to minimize χ^2. This means that n_t of the data points are redundant, since it is (nearly) always possible to fit n_t theoretical parameters to n_t data points. Thus in equation (A.6.15) we should replace N with $N - n_t$. In other words, our condition for a good model for the data in question reads

$$\tilde{\chi}^2 \lesssim 1, \tag{A.6.16}$$

where $\tilde{\chi}^2$ is the *reduced* χ^2, given by

$$\tilde{\chi}^2 = \frac{\chi^2}{N - n_t}. \tag{A.6.17}$$

A.7 Errors On Predictions

Suppose we have obtained $\Phi(T(\mathbf{a})|\mathbf{f}^e)$ for all values of a^α from a fit to some experimental data. We now wish to find what these results tell us about a new physical observable $S(\mathbf{a})$ predicted from the same physical model T. (The index i refers to a given assignment to the set of relevant kinematic observables). The

average value of S with respect to the quantities a^α distributed according to $\Phi(T(\mathbf{a})|\mathbf{f}^e)$ is

$$\langle S\rangle_{\mathbf{f}^e} = \int d\mathbf{a}\Phi(T(\mathbf{a})|\mathbf{f}^e)S(\mathbf{a}). \tag{A.7.1}$$

If we assume that

$$S(\mathbf{a}) = S(\mathbf{a}_0) + \sum_\alpha \left.\frac{\partial S}{\partial a^\alpha}\right|_{\mathbf{a}=\mathbf{a}_0} (a^\alpha - a_0^\alpha) + \frac{1}{2}\sum_{\alpha\beta} \left.\frac{\partial^2 S}{\partial a^\alpha a^\beta}\right|_{\mathbf{a}=\mathbf{a}_0} (a^\alpha - a_0^\alpha)\left(a^\beta - a_0^\beta\right) \tag{A.7.2}$$

and equation (A.5.7) are good approximations over the range of a^α that we wish to investigate, then we find

$$\langle S\rangle_{\mathbf{f}^e} = S(\mathbf{a}_0) + \frac{1}{2}\sum_{\alpha\beta} \left.\frac{\partial^2 S}{\partial a^\alpha a^\beta}\right|_{\mathbf{a}=\mathbf{a}_0} C_{a,\alpha\beta}. \tag{A.7.3}$$

Our uncertainty on S is taken to be the standard deviation $\Delta_{\mathbf{f}^e}S$, where

$$(\Delta_{\mathbf{f}^e}S)^2 = \langle S^2\rangle_{\mathbf{f}^e} - \langle S\rangle_{\mathbf{f}^e}\langle S\rangle_{\mathbf{f}^e}. \tag{A.7.4}$$

Making the same assumptions, we find

$$\Delta_{\mathbf{f}^e}S = \sqrt{\sum_{\alpha\beta} \left.\frac{\partial S}{\partial a^\alpha}\right|_{\mathbf{a}=\mathbf{a}_0} \left.\frac{\partial S}{\partial a^\beta}\right|_{\mathbf{a}=\mathbf{a}_0} C_{a,\alpha\beta}}. \tag{A.7.5}$$

Note in particular that the error on each parameter a^α is

$$\Delta_{\mathbf{f}^e}a^\alpha = C_{a,\alpha\alpha}. \tag{A.7.6}$$

Appendix B

Miscellaneous

In this appendix, we show how certain results derived in this thesis can be obtained when spin is also taken into account. All partons and the proton only have two polarizations, which we write as + and −.

B.1 The DIS Cross Section

Equation (1.2.11) when polarization is taken into account becomes

$$\sigma^{\lambda\lambda'} = \int d^3\mathbf{k}' L^{\mu\nu}_{\lambda}(k,k') \frac{1}{Q^4(P\cdot k)} W^{\lambda'}_{\mu\nu}(P,q), \tag{B.1.1}$$

where λ is the spin of the initial electron, and λ' the spin of the proton. Then the various terms in equation (1.2.11) are

$$\sigma = \frac{1}{4}\sum_{\lambda\lambda'} \sigma^{\lambda\lambda'} \tag{B.1.2}$$

$$L^{\mu\nu} = \frac{1}{2}\sum_{\lambda} L^{\mu\nu}_{\lambda} \tag{B.1.3}$$

$$W_{\mu\nu} = \frac{1}{2}\sum_{\lambda} W^{\lambda}_{\mu\nu}, \tag{B.1.4}$$

and equation (1.2.11) follows from equation (B.1.1)

Let $f_{0i}^{\lambda\lambda'}$ be the PDF of a parton i with spin λ in a proton of spin λ'. Then equation (2.1.3) with spin taken into account is given by

$$W^{\lambda}_{\mu\nu}(P,q) = \sum_{i,\lambda'} \int_x^1 \frac{d\xi}{\xi} W^{i,\lambda'}_{\mu\nu}(p,q) f_{0i}^{\lambda'\lambda}(\xi). \tag{B.1.5}$$

By isotropy,

$$f_{0i}^{++} = f_{0i}^{--} \qquad f_{0i}^{+-} = f_{0i}^{-+}, \tag{B.1.6}$$

so that, averaging equation (B.1.5) over λ, we obtain equation (2.1.3).

B.2 The Plus Prescription

Consider a function $f(x)$ which is singular as $x \to 1$. Its Mellin transform is therefore undefined. However, suppose we add an "endpoint" contribution to $f(x)$ in the form

$$[f(x)]_+ = \lim_{\epsilon\to 0}\left[f(x)\theta(1-x-\epsilon) - \delta(1-x-\epsilon)\left(\int_0^{1-\epsilon} dx' f(x')\right)\right], \tag{B.2.1}$$

This is called the *plus prescription* on $f(x)$. Then in N space we would find

$$\int_0^1 dx x^{N-1}[f(x)]_+ = \int_0^1 dx (x^{N-1}-1) f(x). \tag{B.2.2}$$

For certain functions $f(x)$, equation (B.2.2) is well-defined although the Mellin transform of $f(x)$ is not. Examples of $f(x)$ that fall into this category are

$$f(x) = \frac{\ln^i(1-x)}{1-x} \quad \text{for} \quad i \geq 0. \tag{B.2.3}$$

Appendix C

The $\overline{MS}$ Scheme

C.1 $\overline{MS}$ Renormalization

Consider the calculation of a bare parton initiated cross section $\widehat{\sigma}$, with UV singularities regularized by dimensional regularization, where the number of dimensions $d = 4 - 2\epsilon$ with $\epsilon < 0$. The bare form of a_s, which we write as a_{0s}, is replaced by $a_{0s}\mu^{2\epsilon}$, to keep the dimension of the interaction terms in the Lagrangian of mass dimension d, while ensuring a_{0s} is dimensionless. Note that $a_{0s}\mu^{2\epsilon}$ is defined to be independent of μ^2. Thus $\widehat{\sigma}$ is independent of μ^2. Any collinear singularities are regularized by, for example, giving all partons a small mass. Then, making $\widehat{\sigma}$ dimensionless by multiplying by Q^2 to the suitable power, and normalizing $\widehat{\sigma}$ such that $\widehat{\sigma} = 1$ when $a_{0s} = 0$, we may write the perturbative expansion of $\widehat{\sigma}$ in the form

$$\widehat{\sigma}(Q^2, \epsilon) = 1 + \sum_{n=1}^{\infty} \left(a_{0s} S_\epsilon \left(\frac{Q^2}{\mu^2} \right)^{-\epsilon} \right)^n \widehat{\sigma}'^{(n)}(\epsilon), \tag{C.1.1}$$

where

$$S_\epsilon = \exp[\epsilon(\ln 4\pi - \gamma_E)] \tag{C.1.2}$$

$$\widehat{\sigma}'^{(n)}(\epsilon) = \sum_{m=-n+1}^{\infty} \widehat{\sigma}'^{(n),m}\epsilon^m. \tag{C.1.3}$$

We have pulled out the $\left(S_\epsilon \left(\frac{Q^2}{\mu^2}\right)^{-\epsilon}\right)^n$ from $\widehat{\sigma}'^{(n)}(\epsilon)$, since this term always appears explicitly in calculations.

The relationship between the bare coupling a_{0s} and the renormalized coupling a_s in the $\overline{MS}$ scheme is

$$a_{0s}S_\epsilon \left(\frac{\mu_R^2}{\mu^2}\right)^{-\epsilon} = Z_{a_s}\left(a_s(\mu_R^2,\epsilon),\epsilon\right) a_s(\mu_R^2,\epsilon), \tag{C.1.4}$$

where Z_{a_s} takes the form

$$Z_{a_s}\left(a_s(\mu_R^2,\epsilon),\epsilon\right) = 1 + \sum_{n=1}^{\infty} a_s^n(\mu_R^2,\epsilon) Z_{a_s}^{(n)}(\epsilon), \tag{C.1.5}$$

with

$$Z_{a_s}^{(n)}(\epsilon) = \sum_{m=1}^{n} \frac{Z_{a_s}^{(n),m}}{\epsilon^m}. \tag{C.1.6}$$

This expansion of each of the $Z_{a_s}^{(i)}$ in ϵ guarantees that up to n loops there are enough free parameters $Z_{a_s}^{(i),m}$, $m = 1,...,i$ and $i = 1,...,n$, for us to tune to explicitly remove the UV singularities in the cross section. Furthermore, the number of free parameters $Z_{a_s}^{(i),m}$ we have introduced is the minimum required for this tuning, and hence the subtraction is *minimal*. (This is actually nothing special, since in any scheme the subtraction is minimal once the "redundant" parameters have been specified.)

The $Z_{a_s}^{(i),m}$ are interrelated by one further condition on them, and hence some of them can be eliminated. We will now examine this further condition. Differentiating equation (C.1.4) with respect to $\ln \mu_R^2$ and rearranging, we find

$$\frac{da_s(\mu_R^2,\epsilon)}{d\ln\mu_R^2} = \frac{-\epsilon a_s(\mu_R^2,\epsilon)}{1+\frac{\sum_{n=1}^{\infty} n a_s^n(\mu_R^2,\epsilon) Z_{a_s}^{(n)}(\epsilon)}{Z_{a_s}(a_s(\mu_R^2,\epsilon),\epsilon)}}. \tag{C.1.7}$$

The right hand side of equation (C.1.7) must be non singular as $\epsilon \to 0$. Thus, by expanding equation (C.1.7) in $a_s(\mu_R^2,\epsilon)$, we obtain the form

$$\frac{da_s(\mu_R^2,\epsilon)}{d\ln\mu_R^2} = -\epsilon a_s(\mu_R^2,\epsilon) - \sum_{n=0}^{\infty} \beta_n a_s^{n+2}(\mu_R^2,\epsilon), \tag{C.1.8}$$

where the β_n are independent of ϵ. So, by expanding equation (C.1.7) in $a_s(\mu_R^2,\epsilon)$, we can write some of the $Z_{a_s}^{(n),m}$ in terms of other $Z_{a_s}^{(n),m}$ by demanding equation (C.1.7) is non singular as $\epsilon \to 0$. Once this has been done, then we can use equation (C.1.8) to find the remaining $Z_{a_s}^{(n),m}$ in terms of the β_n. Then the β_n can be obtained simply by demanding that any cross section written as a series in $a_s(\mu_R^2,\epsilon)$ must be finite as $\epsilon \to 0$. (In fact, for certain cross sections not all the β_n can be obtained.)

Equation (C.1.8) can be rewritten

$$s^{-\epsilon}\frac{d(s^\epsilon a_s(s\mu_R^2,\epsilon))}{d\ln s} = \beta(a_s(s\mu_R^2,\epsilon)), \tag{C.1.9}$$

The solution to NLO is

$$s^\epsilon a_s(s\mu_R^2,\epsilon) = a_s(\mu_R^2,\epsilon) + \frac{\beta_0}{\epsilon}(s^{-\epsilon}-1)a_s^2(\mu_R^2,\epsilon) + O(a_s^3(\mu_R^2,\epsilon)). \tag{C.1.10}$$

Continuing to all orders in $a_s(\mu_R^2,\epsilon)$, we find that for $\epsilon < 0$, $a_s(0,\epsilon) = 0$.

C.2 $\overline{MS}$ Factorization

We now regularize both UV and collinear singularities using dimensional regularization, and assume $Z_{a_s}(a_s,\epsilon)$ has been calculated to the appropriate order.

In $\hat{\sigma}$, we write a_{0s} in terms of a_s as in equation (C.1.4), so that all UV divergences vanish. Then $\hat{\sigma}$ takes the form

$$\hat{\sigma}(a_s(Q^2,\epsilon),\epsilon) = 1 + \sum_{n=1}^{\infty} a_s^n(Q^2,\epsilon)\hat{\sigma}^{(n)}(\epsilon), \tag{C.2.11}$$

where

$$\hat{\sigma}^{(n)}(\epsilon) = \sum_{m=-n}^{\infty} \hat{\sigma}^{(n),m}\epsilon^m. \tag{C.2.12}$$

The singularities as $\epsilon \to 0$ in $\hat{\sigma}$ are now purely collinear singularities. From the factorization theorem, the relationship between the bare partonic cross section $\hat{\sigma}$ and the factorized partonic cross section $\tilde{\sigma}$ is

$$\hat{\sigma}^{[a]}(a_s(Q^2,\epsilon),\epsilon) = O\{\tilde{\sigma}^{[a]}(a_s(Q^2,\epsilon), a_s(\mu_F^2,\epsilon),\epsilon)\Gamma^a(a_s(\mu_F^2,\epsilon),\epsilon)\}, \tag{C.2.13}$$

and $\tilde{\sigma}$ is non singular as $\epsilon \to 0$. a refers to the number of partons in the initial state. The O refers to any relevant ordering of the objects $\tilde{\sigma}$ and the Γ. Note in the singlet case that the objects $\hat{\sigma}$ and $\tilde{\sigma}$ have indices, where the number of indices is equal to the number of incoming partons. Γ is an object with two indices. Thus in the singlet case equation (C.2.13) involves contraction over indices. In the non singlet case, these objects are just single numbers. Γ in the $\overline{MS}$ factorization scheme takes the form

$$\Gamma(a_s(\mu_F^2,\epsilon),\epsilon) = 1 + \sum_{n=1}^{\infty} a_s^n(\mu_F^2,\epsilon)\Gamma^{(n)}(\epsilon), \tag{C.2.14}$$

where

$$\Gamma^{(n)}(\epsilon) = \sum_{m=1}^{n} \frac{\Gamma^{(n),n}}{\epsilon^m}. \tag{C.2.15}$$

Now

$$\frac{d\Gamma(a_s,\epsilon)}{d\ln\mu_F^2}\Gamma^{-1}(a_s,\epsilon) = \left(-\epsilon a_s - \sum_{n=0}^{\infty}\beta_n a_s^{n+2}\right)$$

$$\times\left(\sum_{n=1}^{\infty} n a_s^{n-1}\Gamma^{(n)}(\epsilon)\right)\left(1+\sum_{n=1}^{\infty} a_s^n\Gamma^{(n)}(\epsilon)\right)^{-1}. \qquad \text{(C.2.16)}$$

From the factorization theorem, this equation is non singular as $\epsilon \to 0$. Expanding equation (C.2.16) in a_s therefore gives the form

$$\frac{d\Gamma(a_s,\epsilon)}{d\ln\mu_F^2}\Gamma^{-1}(a_s,\epsilon) = a_s\sum_{n=0}^{\infty} P^{(n)}a_s^n, \qquad \text{(C.2.17)}$$

where the $P^{(n)}$ are independent of ϵ. So, by expanding equation (C.2.16) in a_s, we can write some of the $\Gamma^{(n),m}$ in terms of other $\Gamma^{(n),m}$ by demanding equation (C.2.16) is non singular as $\epsilon \to 0$. Once this has been done, then expanding equation (C.2.17) in a_s we can find the remaining $\Gamma^{(n),m}$ in terms of the $P^{(n)}$. Then the $P^{(n)}$ can be explicitly obtained simply by demanding that the factorized partonic cross section written as a series in $a_s(\mu_R^2,\epsilon)$ must be finite as $\epsilon \to 0$. (In fact, for certain cross sections not all the $P^{(n)}$ can be obtained.)

The solution to equation (C.2.17), with the boundary condition $\Gamma(0,\epsilon) = 1$ which follows from equation (C.2.14), is

$$\Gamma(a_s,\epsilon) = P\exp\left[\int_0^{a_s} dr\frac{rP(r)}{\beta(r)-\epsilon r}\right]. \qquad \text{(C.2.18)}$$

(The path ordering P is unnecessary in the non singlet case.) Since $a_s(0,\epsilon) = 0$, we can write Γ in the alternative form

$$\Gamma(a_s(\mu_F^2,\epsilon),\epsilon) = P\exp\left[\int_0^{\mu_F^2}\frac{dq^2}{q^2}a_s(q^2,\epsilon)P(a_s(q^2,\epsilon))\right]. \qquad \text{(C.2.19)}$$

Appendix D

Results For Soft Resummation

D.1 PDF Parameters

Parameter	Value	Error
a_Σ	-1.15765	0.00983
b_Σ	3.58987	0.02325
c_Σ	8.05251	0.42397
d_Σ	-1.86265	0.21191
A_Σ	0.53986	0.01296
a_g	-0.90248	0.04552
b_g	19.92489	0.89609
c_g	48.29034	4.20934
d_g	-10.57826	0.74558
A_3	0.10347	0.00988
a_3	2.18193	0.18023
b_3	5.92008	0.12897
c_3	1.98698	0.03033
d_3	-2.60866	0.02171
A_8	0.47852	0.04256
a_8	1.17924	0.10344
b_8	8.18339	0.15448
c_8	2.73360	0.04224
d_8	-2.94519	0.02473
A_H	-44.98646	11.23403
a_H	1.96194	0.07448
b_H	34.03576	1.03480
c_H	-1062.44756	222.13483
d_H	90.49333	63.42495

Table D.1.1: Singlet and non singlet PDF fits with resummation for $k_R = 1, k_F = 1$. $\tilde{\chi}^2 = 0.9310$

Parameter	Value	Error
a_{d^-}	0.37463	0.05599
b_{d^-}	5.61679	0.19893
a_{u^-}	-0.46965	0.02143
b_{u^-}	3.19291	0.09603
A_{H_3}	-1.77437	6.95398
a_{H_3}	1.47387	1.84582
b_{H_3}	0.44912	4.33520

Table D.1.2: Valence quark PDF fits with resummation for $k_R = 1, k_F = 1$. $\tilde{\chi}^2 = 0.7630$

Parameter	Value	Error
a_Σ	-1.17060	0.00833
b_Σ	3.57961	0.02559
c_Σ	4.22011	0.21852
d_Σ	-0.89789	0.20982
A_Σ	0.51936	0.01119
a_g	-0.95518	0.06297
b_g	25.56836	1.85362
c_g	40.86768	3.86063
d_g	-10.08883	0.77153
A_3	0.08760	0.00617
a_3	2.20938	0.15180
b_3	6.17501	0.13276
c_3	2.02320	0.04206
d_3	-2.66124	0.03169
A_8	0.36956	0.03395
a_8	1.31584	0.08630
b_8	7.92842	0.14510
c_8	2.50898	0.06086
d_8	-2.82100	0.04185
A_H	6.98437	4.42244
a_H	1.09066	0.23720
b_H	14.74509	1.32048
c_H	-139.73943	10.31829
d_H	51.75185	4.04146

Table D.1.3: Singlet and non singlet PDF fits with resummation for $k_R = 4, k_F = 4$. $\widetilde{\chi}^2 = 0.9540$

Parameter	Value	Error
a_{d^-}	0.26584	0.05580
b_{d^-}	5.91766	0.20761
a_{u^-}	-0.53109	0.02041
b_{u^-}	3.30239	0.08935
A_{H_3}	-229.97660	80.92541
a_{H_3}	2.78583	1.65418
b_{H_3}	9.34261	6.15772

Table D.1.4: Valence quark PDF fits with resummation for $k_R = 4, k_F = 4$. $\widetilde{\chi}^2 = 0.7610$

Parameter	Value	Error
a_Σ	-1.33178	0.00263
b_Σ	5.33856	0.00743
c_Σ	124.65229	1.30137
d_Σ	-18.80549	0.16367
A_Σ	0.60138	0.00103
a_g	-1.57971	0.00232
b_g	11.15423	0.05041
c_g	-9222.45178	192.67720
d_g	-982.08919	19.88245
A_3	0.11193	0.02382
a_3	2.59042	0.01747
b_3	5.86980	0.01426
c_3	1.96643	0.00586
d_3	-2.63638	0.00619
A_8	0.59427	0.00167
a_8	2.62312	0.00527
b_8	3.90424	0.05687
c_8	1.12179	0.00065
d_8	-2.05407	0.00058
A_H	2.12044	0.03733
a_H	-0.02918	0.00293
b_H	0.05950	0.03503
c_H	3.47842	0.01463
d_H	-3.67641	0.00976

Table D.1.5: Singlet and non singlet PDF fits with resummation for $k_R = 4, k_F = \frac{1}{4}$. $\widetilde{\chi}^2 = 0.9200$

Parameter	Value	Error
a_{d^-}	0.56524	0.04799
b_{d^-}	5.62104	0.18018
a_{u^-}	-0.49799	0.01863
b_{u^-}	2.86789	0.09137
A_{H_3}	1370.79878	93.02141
a_{H_3}	6.47124	0.88177
b_{H_3}	3.88028	1.04585

Table D.1.6: Valence quark PDF fits with resummation for $k_R = 4, k_F = \frac{1}{4}$. $\widetilde{\chi}^2 = 0.7800$

Parameter	Value	Error
a_Σ	-1.06520	0.01667
b_Σ	5.58180	0.02834
c_Σ	-5416.15308	709.30969
d_Σ	-1894.97784	247.09109
A_Σ	0.49778	0.00611
a_g	-0.87599	0.03097
b_g	8.46643	0.19552
c_g	-20.18210	0.35805
d_g	7.02017	0.15552
A_3	0.07379	0.00886
a_3	1.97244	0.24607
b_3	6.69805	0.19841
c_3	2.08948	0.03197
d_3	-2.67939	0.02158
A_8	0.35697	0.01458
a_8	-1.16612	0.01120
b_8	23.84851	0.68415
c_8	33.28751	1.22460
d_8	-7.37707	0.27237
A_H	-11.75001	0.55149
a_H	0.88881	0.02996
b_H	-1.94524	0.03233
c_H	1.46048	0.00282
d_H	-2.40557	0.00241

Table D.1.7: Singlet and non singlet PDF fits with resummation for $k_R = \frac{1}{4}, k_F = 4$. $\tilde{\chi}^2 = 1.1440$

Parameter	Value	Error
a_{d^-}	0.18347	0.01370
b_{d^-}	6.32688	0.14827
a_{u^-}	-0.52950	0.00918
b_{u^-}	3.56288	0.03883
A_{H_3}	16.16540	1.70255
a_{H_3}	0.72074	0.06115
b_{H_3}	10.03756	1.43654

Table D.1.8: Valence quark PDF fits with resummation for $k_R = \frac{1}{4}, k_F = 4$. $\tilde{\chi}^2 = 0.9220$

Parameter	Value	Error
a_Σ	-1.25180	0.00777
b_Σ	3.08645	0.02031
c_Σ	10.07754	0.54725
d_Σ	0.19364	0.26146
A_Σ	0.41366	0.00898
a_g	-0.37909	0.05406
b_g	0.76962	0.06376
c_g	1.98725	0.01613
d_g	-2.76371	0.01204
A_3	0.13472	0.01276
a_3	2.16617	0.16356
b_3	5.59495	0.11756
c_3	1.98040	0.04913
d_3	-2.51758	0.03904
A_8	0.31056	0.02543
a_8	1.23978	0.16131
b_8	7.15427	0.20870
c_8	125.15048	137.52059
d_8	116.66939	137.15970
A_H	9.87298	3.18867
a_H	2.77134	0.08708
b_H	8.99594	0.29468
c_H	37.18121	24.80182
d_H	37.81914	14.03714

Table D.1.9: Singlet and non singlet PDF fits with resummation for $k_R = \frac{1}{4}, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 1.0290$

Parameter	Value	Error
a_{d^-}	0.75822	0.08673
b_{d^-}	5.61234	0.25688
a_{u^-}	-0.39285	0.02736
b_{u^-}	2.85467	0.09781
A_{H_3}	16.67783	39.17209
a_{H_3}	1.14898	0.76246
b_{H_3}	7.22367	5.63383

Table D.1.10: Valence quark PDF fits with resummation for $k_R = \frac{1}{4}, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 0.8150$

Parameter	Value	Error
a_Σ	-1.32035	0.00883
b_Σ	3.07762	0.02029
c_Σ	7.33948	0.82590
d_Σ	9.35880	0.99491
A_Σ	0.45820	0.00975
a_g	-0.83084	0.03258
b_g	4.45501	0.68864
c_g	2.45339	0.08484
d_g	-2.97432	0.04697
A_3	0.10885	0.00875
a_3	2.03398	0.13098
b_3	5.38956	0.09024
c_3	1.91684	0.02457
d_3	-2.57078	0.01873
A_8	0.27976	0.03430
a_8	1.51408	0.09983
b_8	7.24110	0.13306
c_8	2.80514	0.16325
d_8	-2.62865	0.13998
A_H	2.20748	0.80448
a_H	4.88014	0.32860
b_H	0.87598	0.12485
c_H	2.50347	3.14438
d_H	6.22038	2.93198

Table D.1.11: Singlet and non singlet PDF fits without resummation for $k_R = 1, k_F = 1$. $\tilde{\chi}^2 = 0.9090$

Parameter	Value	Error
a_{d^-}	0.37182	0.06224
b_{d^-}	5.57672	0.21074
a_{u^-}	-0.48851	0.02164
b_{u^-}	2.97949	0.08402
A_{H_3}	-1650.70000	81.71676
a_{H_3}	3.57894	1.48762
b_{H_3}	11.04765	5.36836

Table D.1.12: Valence quark PDF fits without resummation for $k_R = 1, k_F = 1$. $\tilde{\chi}^2 = 0.7840$

Parameter	Value	Error
a_Σ	-1.19920	0.00482
b_Σ	3.09092	0.01691
c_Σ	1.01165	0.09756
d_Σ	0.79738	0.10736
A_Σ	0.51829	0.00621
a_g	-1.16778	0.03194
b_g	6.37794	0.99860
c_g	1.24906	0.52704
d_g	-1.49421	0.25443
A_3	0.09757	0.00719
a_3	1.64930	0.09033
b_3	5.30761	0.06943
c_3	1.96918	0.02780
d_3	-2.56028	0.02185
A_8	0.38702	0.02082
a_8	0.03555	0.06034
b_8	5.60324	0.08817
c_8	11836.30250	5487.17627
d_8	4883.46708	2451.32086
A_H	4.07270	0.83449
a_H	3.84495	0.14472
b_H	-0.01958	0.08627
c_H	-3.26591	0.38572
d_H	3.69002	0.34364

Table D.1.13: Singlet and non singlet PDF fits without resummation for $k_R = 4, k_F = 4$. $\tilde{\chi}^2 = 0.9110$

Parameter	Value	Error
a_{d^-}	0.23608	0.05591
b_{d^-}	5.66278	0.19710
a_{u^-}	-0.54427	0.02045
b_{u^-}	3.07268	0.08365
A_{H_3}	-373.87309	92.19010
a_{H_3}	2.63747	0.98564
b_{H_3}	9.94245	4.04856

Table D.1.14: Valence quark PDF fits without resummation for $k_R = 4, k_F = 4$. $\tilde{\chi}^2 = 0.8390$

Parameter	Value	Error
a_Σ	-1.49607	0.00720
b_Σ	3.03874	0.01452
c_Σ	93.69061	6.04141
d_Σ	-3.14263	0.81486
A_Σ	0.40804	0.00816
a_g	-1.94055	0.00841
b_g	9.19797	0.43351
c_g	-1466.45680	357.99333
d_g	-22.94967	7.00107
A_3	0.12064	0.00827
a_3	2.20346	0.09847
b_3	5.31107	0.07269
c_3	1.92784	0.02198
d_3	-2.58774	0.01708
A_8	0.05281	0.00894
a_8	2.43061	0.18829
b_8	5.08162	0.15275
c_8	-4.42809	0.05200
d_8	1.91104	0.03867
A_H	1.37613	0.45962
a_H	4.17711	0.19695
b_H	1.16443	0.10588
c_H	-1.82725	4.38219
d_H	17.96501	3.70133

Table D.1.15: Singlet and non singlet PDF fits without resummation for $k_R = 4, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 0.9150$

Parameter	Value	Error
a_{d^-}	0.52160	0.07435
b_{d^-}	5.48305	0.23268
a_{u^-}	-0.48245	0.02363
b_{u^-}	2.81345	0.09138
A_{H_3}	-27.44339	73.45846
a_{H_3}	1.62636	1.09819
b_{H_3}	6.95172	4.44441

Table D.1.16: Valence quark PDF fits without resummation for $k_R = 4, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 0.8210$

Parameter	Value	Error
a_Σ	-1.16639	0.00000
b_Σ	16.91278	0.00005
c_Σ	-1.64960	0.00000
d_Σ	-0.04641	0.00001
A_Σ	0.84020	0.00000
a_g	-1.99985	0.00000
b_g	0.52504	0.00000
c_g	-4221.29424	0.00116
d_g	2876.69076	0.00079
A_3	0.10136	0.00136
a_3	1.48244	0.00902
b_3	5.46011	0.01249
c_3	2.04327	0.00427
d_3	-2.62149	0.00389
A_8	1.81619	0.00002
a_8	-0.15356	0.00000
b_8	1.94580	0.00001
c_8	1.00015	0.00000
d_8	-1.98596	0.00001
A_H	-50.40890	0.58146
a_H	0.46279	0.00244
b_H	7.69147	0.05995
c_H	5.92529	0.00850
d_H	-4.73274	0.00365

Table D.1.17: Singlet and non singlet PDF fits without resummation for $k_R = \frac{1}{4}, k_F = 4$. $\widetilde{\chi}^2 = 1.0300$

Parameter	Value	Error
a_{d^-}	-0.03091	0.06141
b_{d^-}	5.36853	0.23273
a_{u^-}	-0.48506	0.02961
b_{u^-}	3.50334	0.12399
A_{H_3}	0.01706	0.01980
a_{H_3}	-1.03468	0.29973
b_{H_3}	-3.91489	0.93907

Table D.1.18: Valence quark PDF fits without resummation for $k_R = \frac{1}{4}, k_F = 4$. $\widetilde{\chi}^2 = 0.9310$

Parameter	Value	Error
a_Σ	-1.25097	0.00533
b_Σ	3.11815	0.01846
c_Σ	15.87796	0.53152
d_Σ	-1.16463	0.19664
A_Σ	0.36284	0.01125
a_g	0.31623	0.05590
b_g	4.61825	0.18088
c_g	3.05950	0.01584
d_g	-3.46981	0.00948
A_3	0.13241	0.00755
a_3	2.53979	0.07961
b_3	5.36106	0.06398
c_3	1.85385	0.01674
d_3	-2.55667	0.01345
A_8	0.25350	0.02427
a_8	1.95853	0.08636
b_8	8.48707	0.15591
c_8	3.16538	0.03827
d_8	-3.33709	0.02193
A_H	0.06960	0.02803
a_H	-0.26602	0.05028
b_H	-1.81881	1.61134
c_H	-1.23585	0.52797
d_H	-0.02342	0.79431

Table D.1.19: Singlet and non singlet PDF fits without resummatio n for $k_R = \frac{1}{4}, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 0.9350$

Parameter	Value	Error
a_{d^-}	0.72165	0.07929
b_{d^-}	5.57119	0.23359
a_{u^-}	-0.39348	0.02776
b_{u^-}	2.74398	0.08963
A_{H_3}	37.67570	34.20754
a_{H_3}	1.30823	1.06675
b_{H_3}	13.23538	9.77882

Table D.1.20: Valence quark PDF fits without resummation for $k_R = \frac{1}{4}, k_F = \frac{1}{4}$. $\tilde{\chi}^2 = 0.7770$

D.2 Plots Of Higher Twist Contributions to F_2

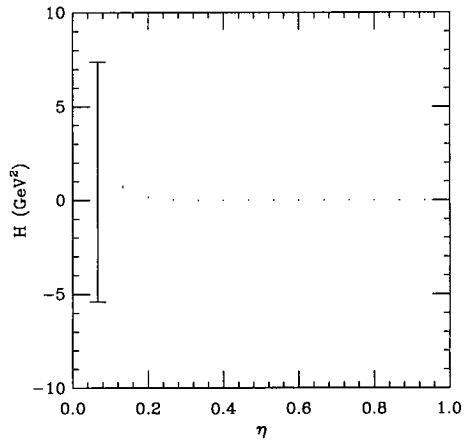

Figure D.2.1: Higher twist fit with resummation, $k_R = 1$, $k_F = 1$.

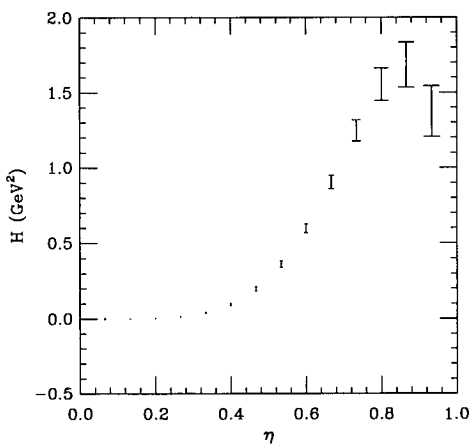

Figure D.2.2: Higher twist fit without resummation, $k_R = 1$, $k_F = 1$.

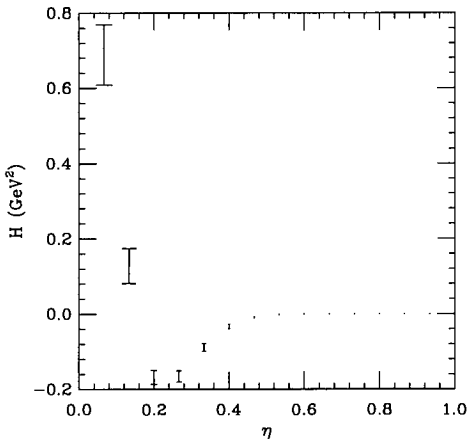

Figure D.2.3: Higher twist fit with resummation, $k_R = 4$, $k_F = 4$.

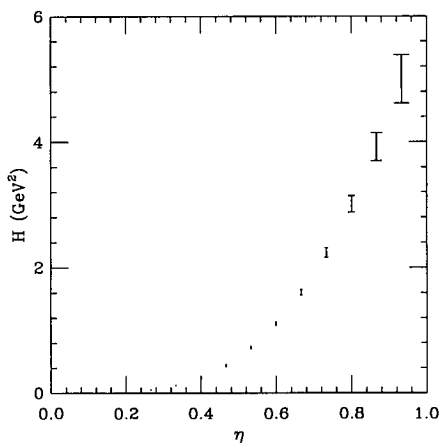

Figure D.2.4: Higher twist fit without resummation, $k_R = 4$, $k_F = 4$.

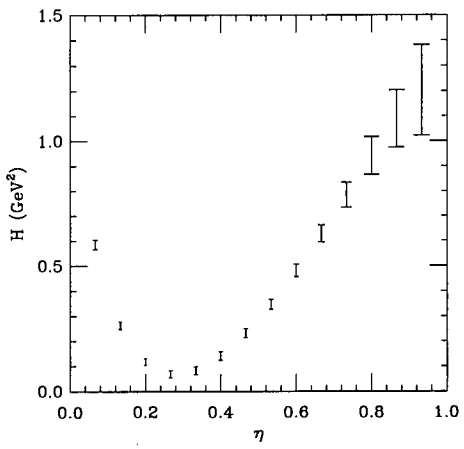

Figure D.2.5: Higher twist fit with resummation, $k_R = 4$, $k_F = \frac{1}{4}$.

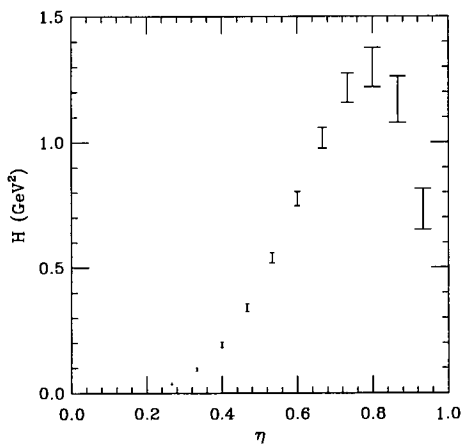

Figure D.2.6: Higher twist fit without resummation, $k_R = 4$, $k_F = \frac{1}{4}$.

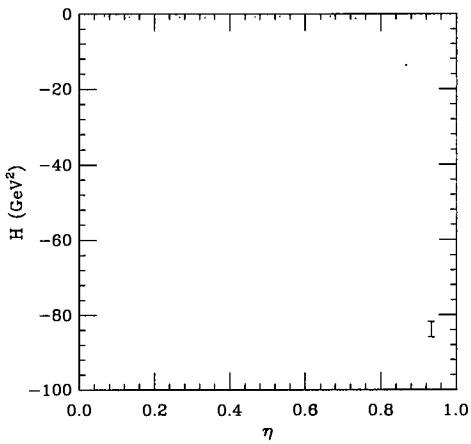

Figure D.2.7: Higher twist fit with resummation, $k_R = \frac{1}{4}$, $k_F = 4$.

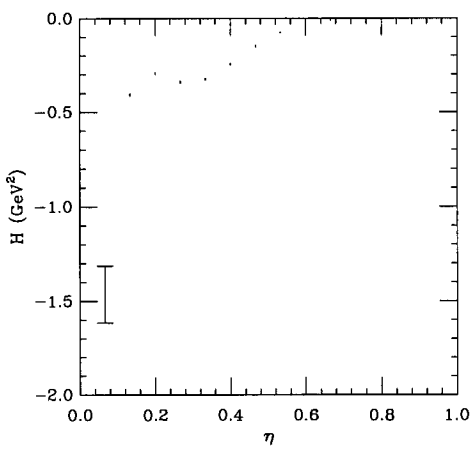

Figure D.2.8: Higher twist fit without resummation, $k_R = \frac{1}{4}$, $k_F = 4$.

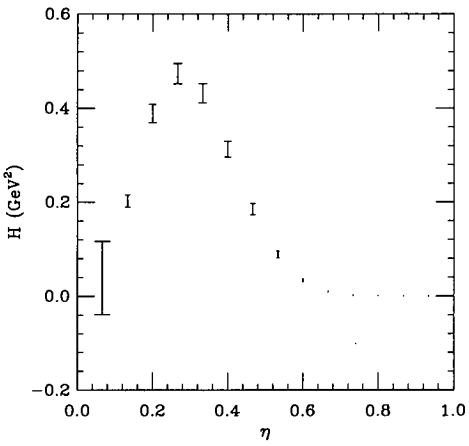

Figure D.2.9: Higher twist fit with resummation, $k_R = \frac{1}{4}$, $k_F = \frac{1}{4}$.

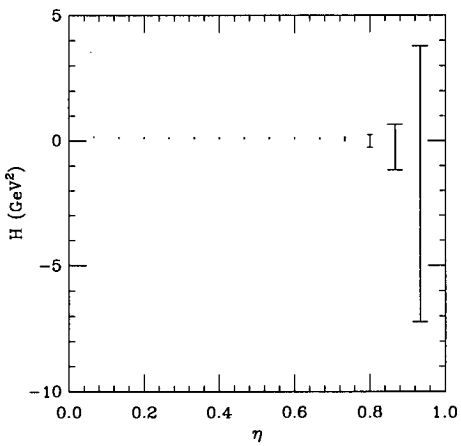

Figure D.2.10: Higher twist fit without resummation, $k_R = \frac{1}{4}$, $k_F = \frac{1}{4}$.

D.3 Plots Of Higher Twist Contributions to F_3

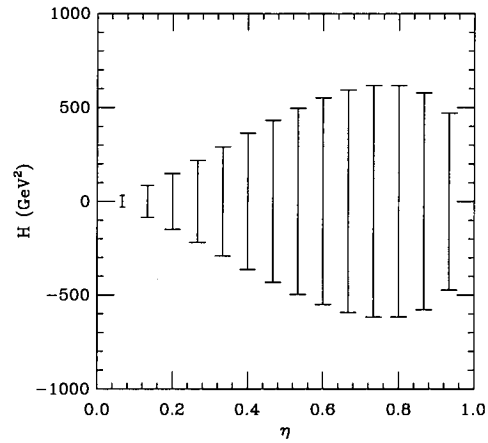

Figure D.3.1: Higher twist fit with resummation, $k_R = 1$, $k_F = 1$.

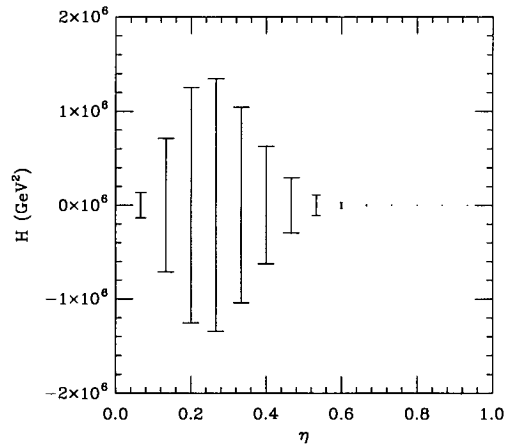

Figure D.3.2: Higher twist fit without resummation, $k_R = 1$, $k_F = 1$.

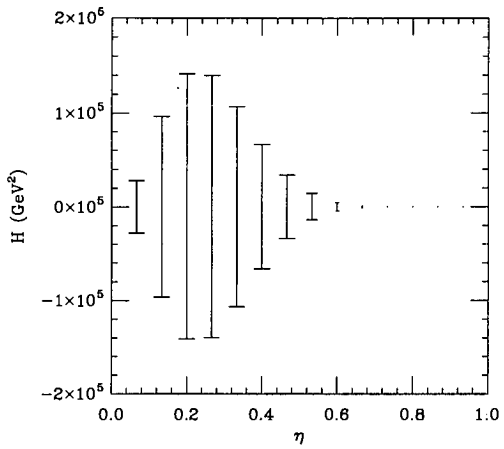

Figure D.3.3: Higher twist fit with resummation, $k_R = 4$, $k_F = 4$.

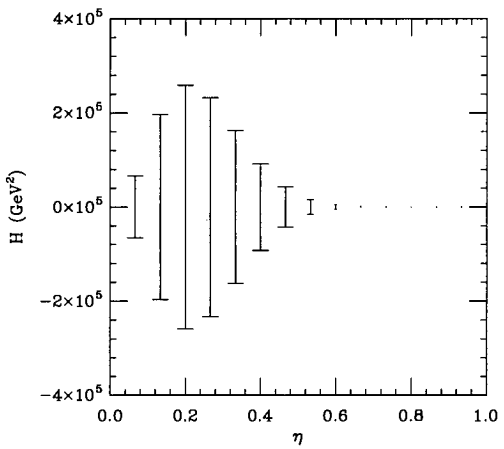

Figure D.3.4: Higher twist fit without resummation, $k_R = 4$, $k_F = 4$.

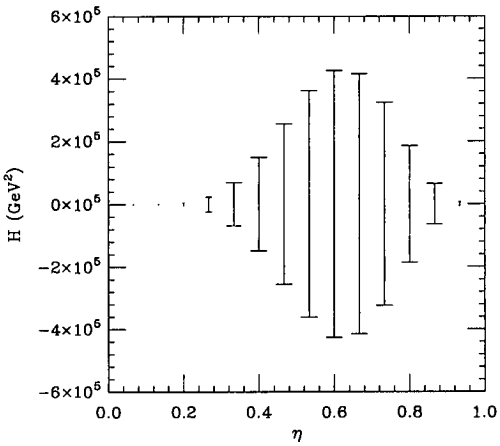

Figure D.3.5: Higher twist fit with resummation, $k_R = 4$, $k_F = \frac{1}{4}$.

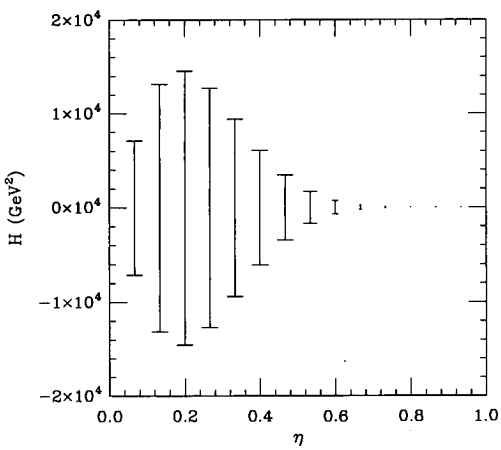

Figure D.3.6: Higher twist fit without resummation, $k_R = 4$, $k_F = \frac{1}{4}$.

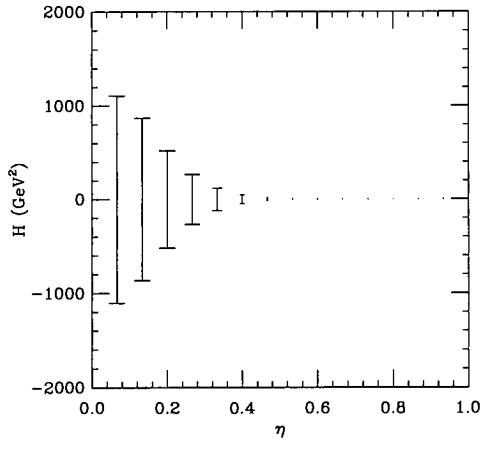

Figure D.3.7: Higher twist fit with resummation, $k_R = \frac{1}{4}$, $k_F = 4$.

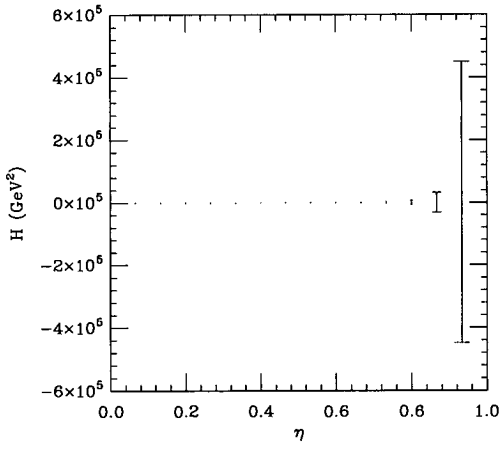

Figure D.3.8: Higher twist fit without resummation, $k_R = \frac{1}{4}$, $k_F = 4$.

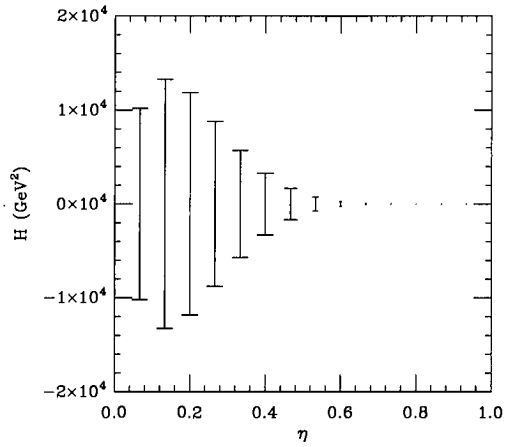

Figure D.3.9: Higher twist fit with resummation, $k_R = \frac{1}{4}$, $k_F = \frac{1}{4}$.

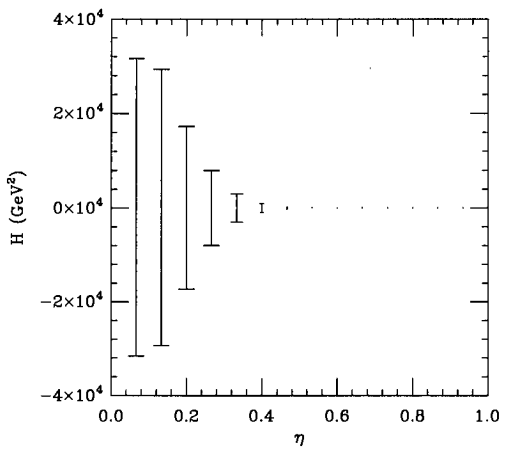

Figure D.3.10: Higher twist fit without resummation, $k_R = \frac{1}{4}$, $k_F = \frac{1}{4}$.

D.4 Drell-Yan Predictions Using MRST99 PDF's

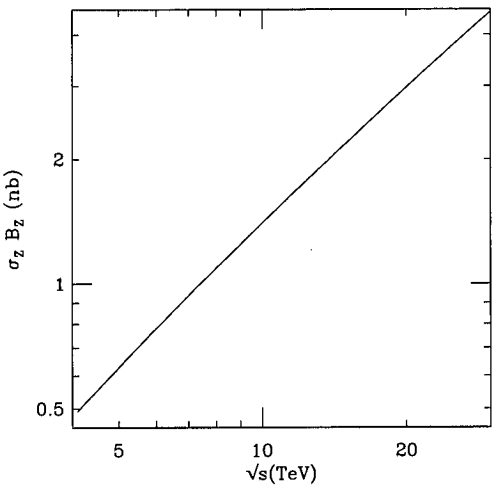

Figure D.4.1: pp Z production.

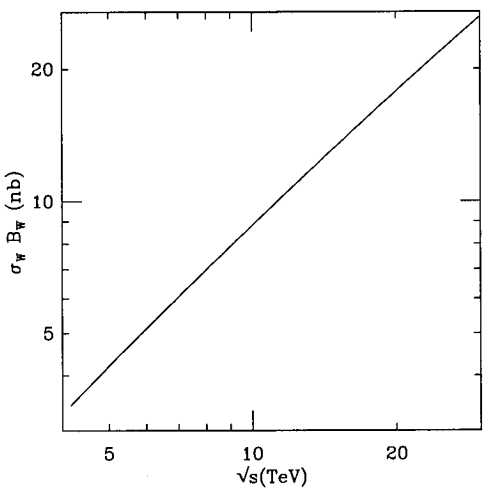

Figure D.4.2: pp W^+ production.

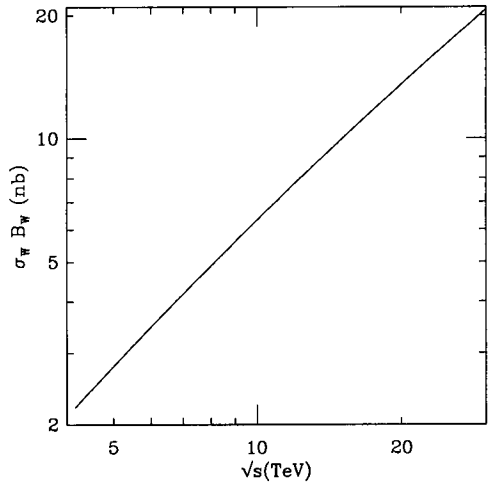

Figure D.4.3: pp W^- production.

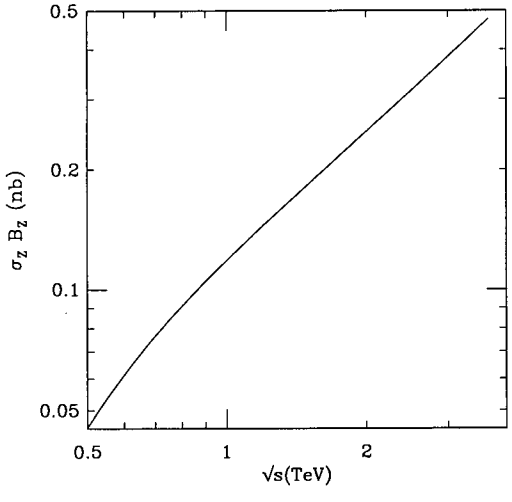

Figure D.4.4: $p\bar{p}$ Z production.

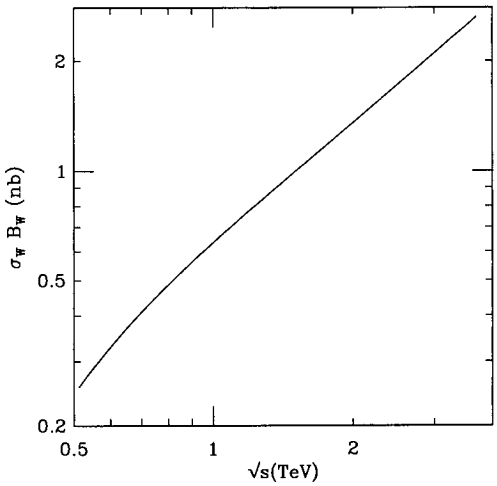

Figure D.4.5: $p\bar{p}$ $W^{+/-}$ production.

D.5 Drell-Yan Predictions versus $\sqrt{s}$

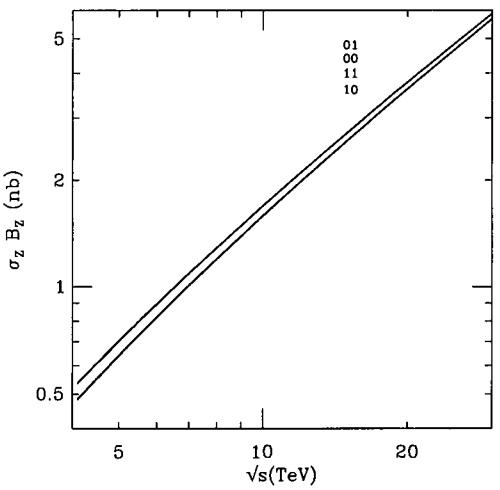

Figure D.5.1: pp Z production.

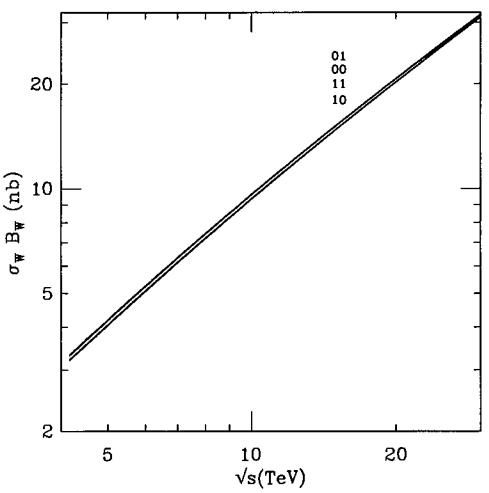

Figure D.5.2: pp W^+ production.

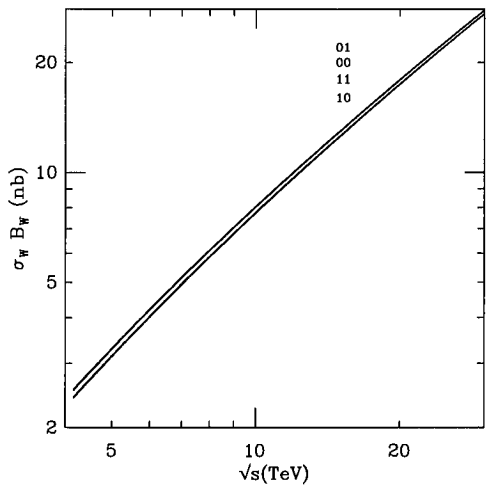

Figure D.5.3: pp W^- production.

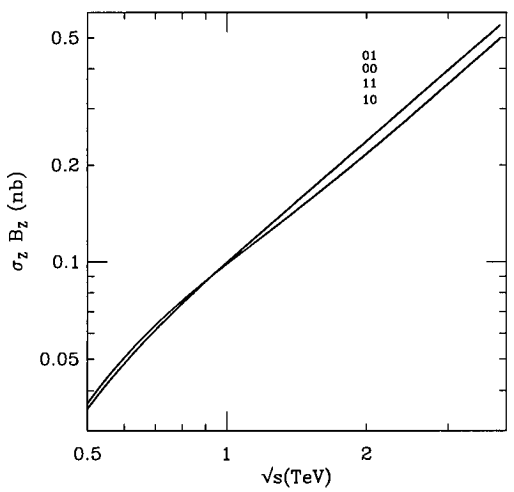

Figure D.5.4: $p\bar{p}$ Z production.

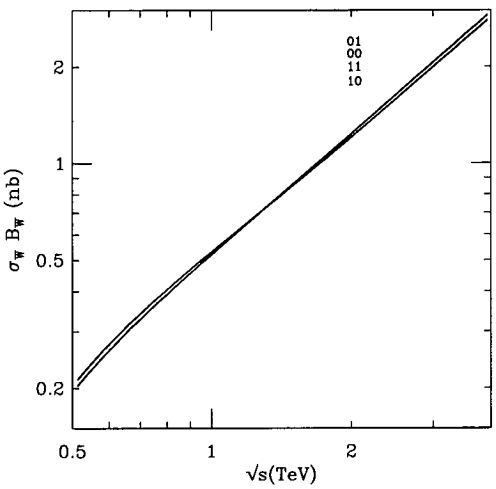

Figure D.5.5: $p\bar{p}$ $W^{+/-}$ production.

D.6 Drell-Yan Predictions With And Without Resummation

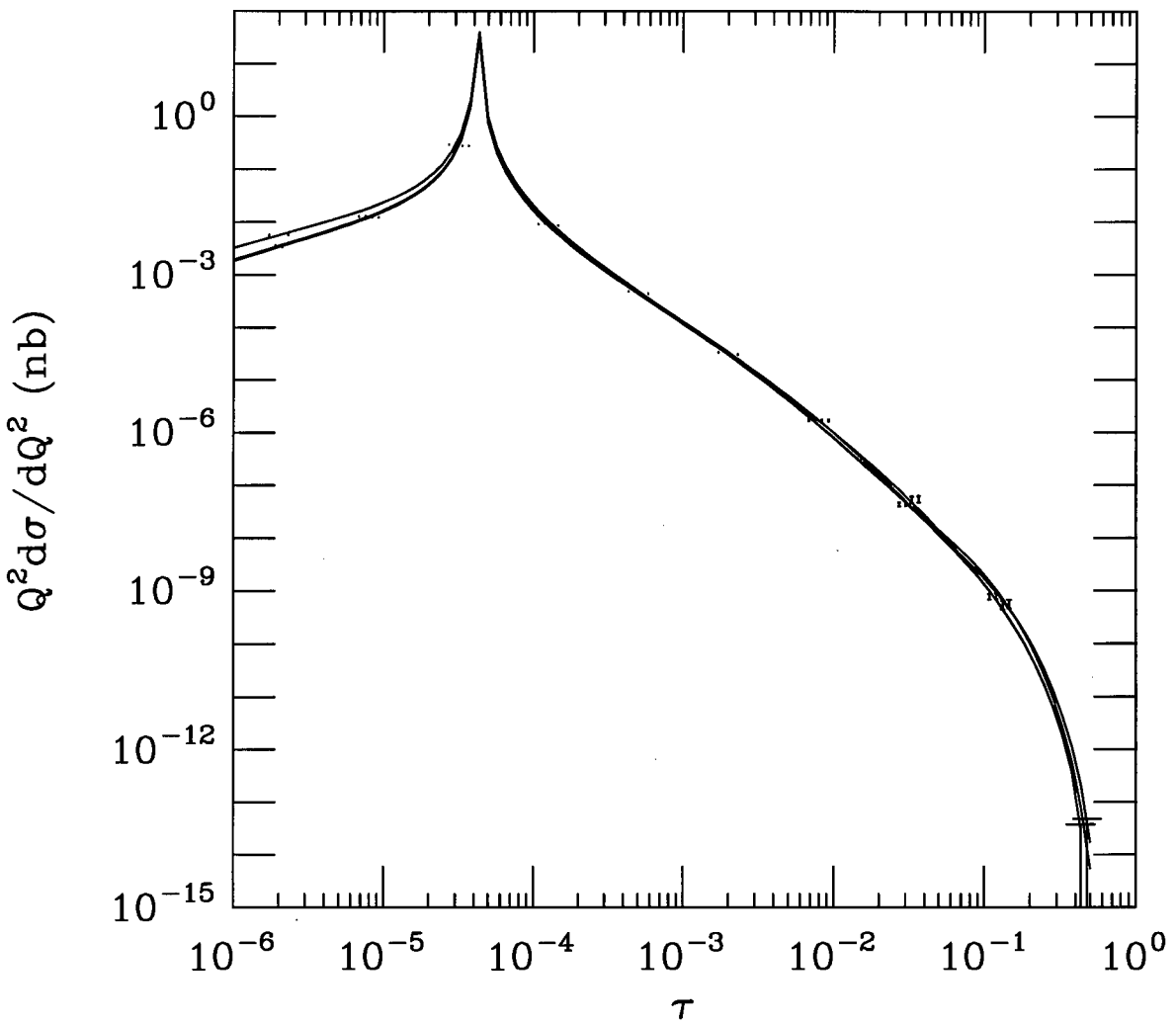

Figure D.6.1: LHC Z production.

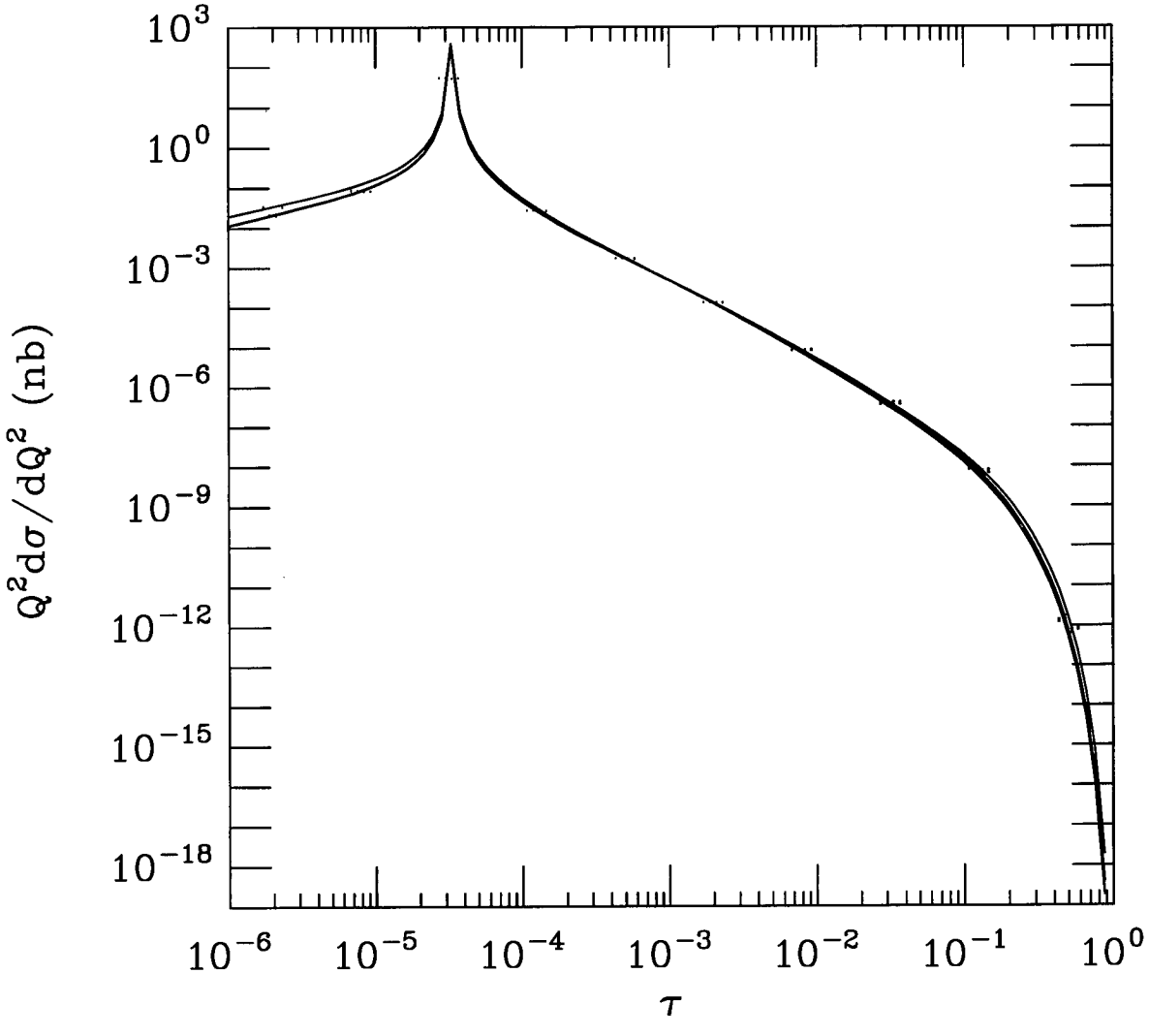

Figure D.6.2: LHC W^+ production.

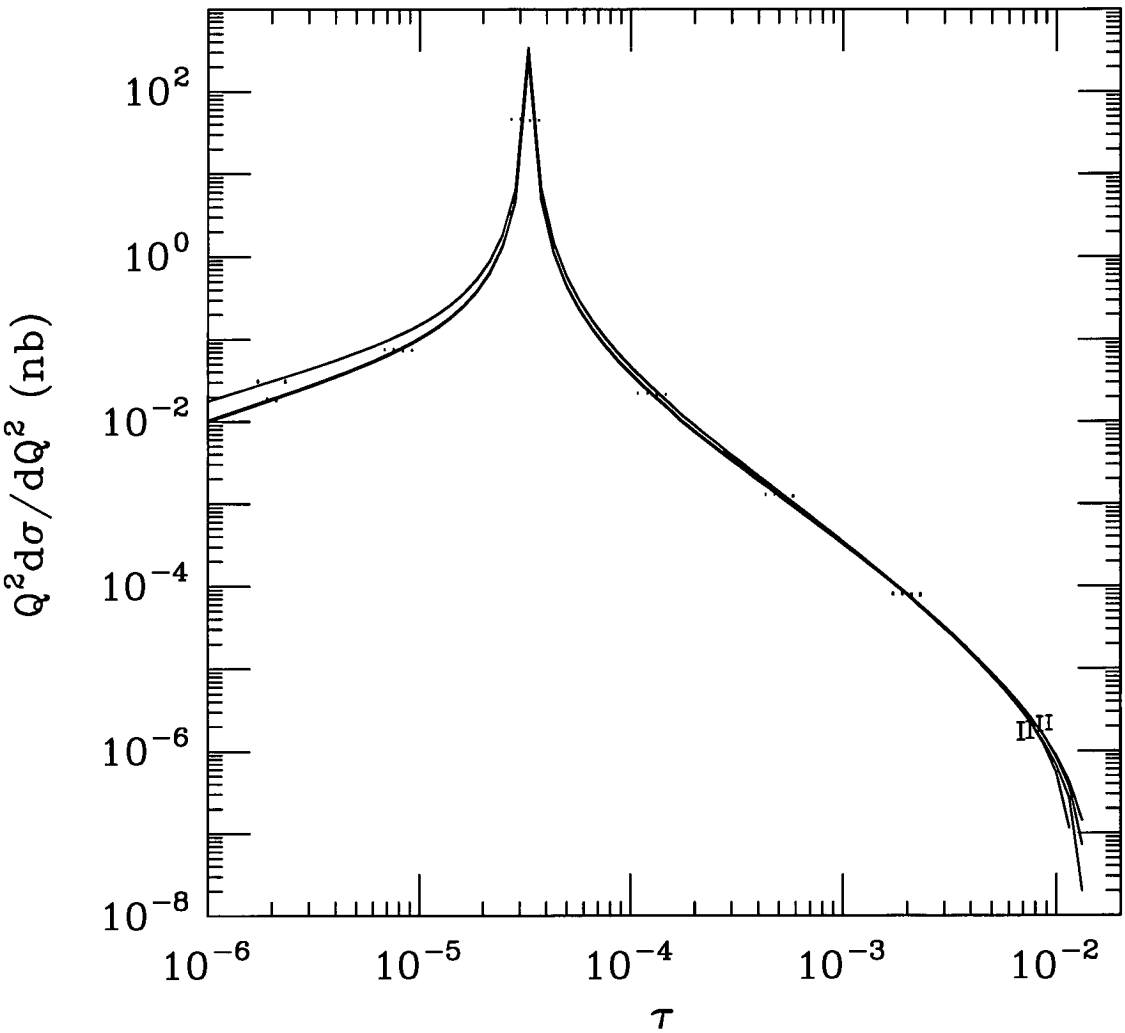

Figure D.6.3: LHC W^- production.

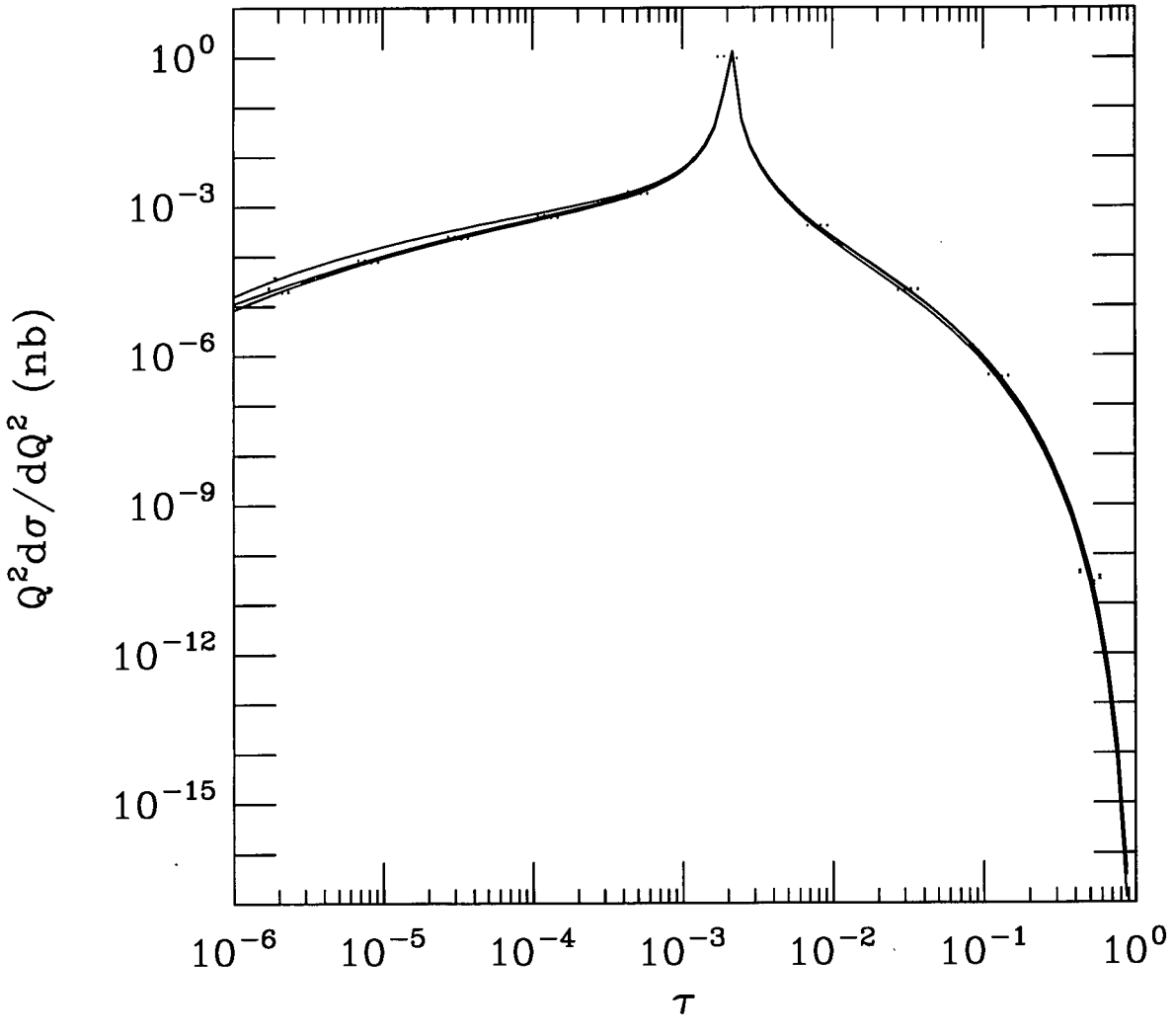

Figure D.6.4: TEVATRON Z production.

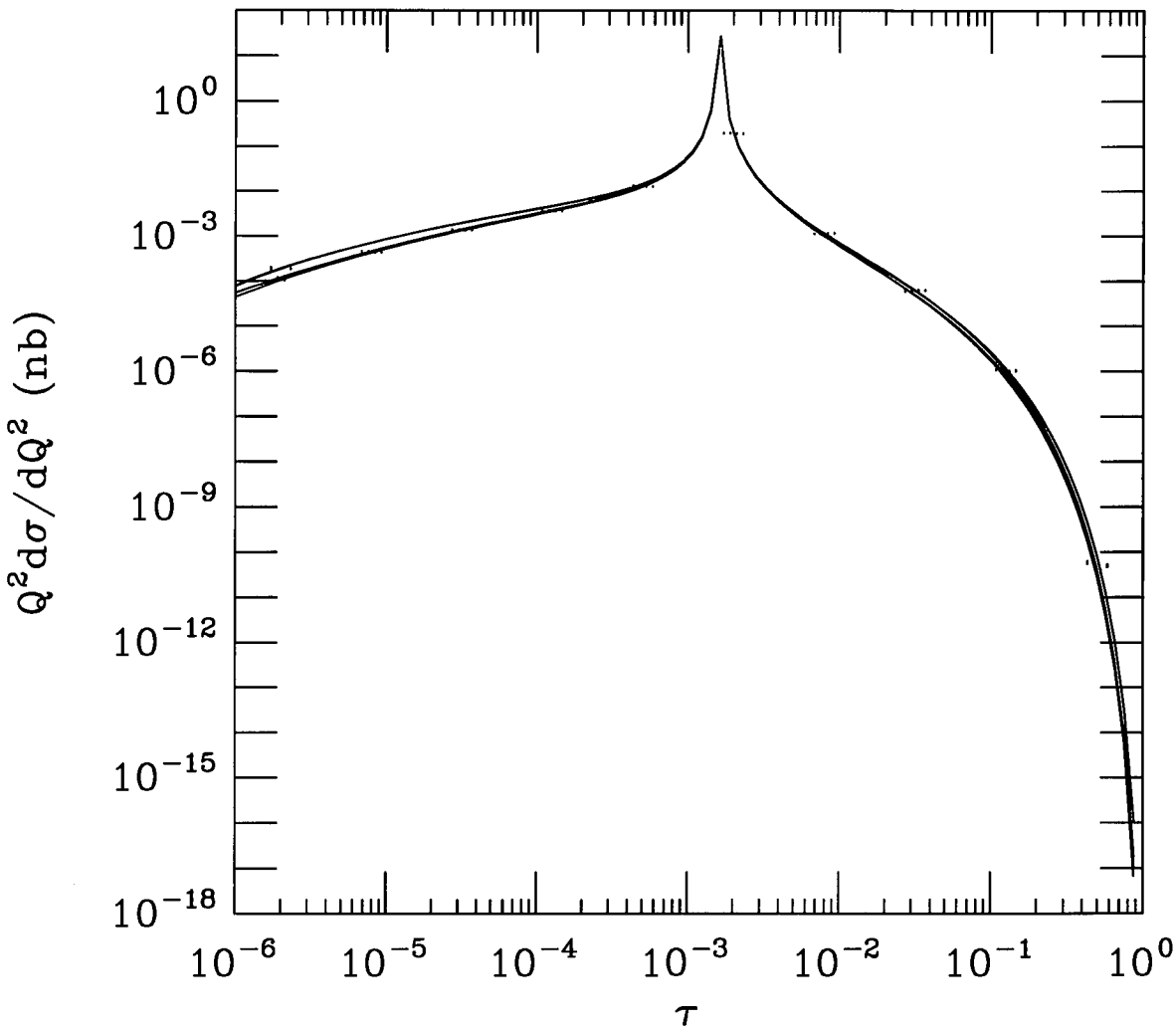

Figure D.6.5: TEVATRON $W^{+/-}$ production.

D.7 Drell-Yan Cross Section Ratios With And Without Resummation

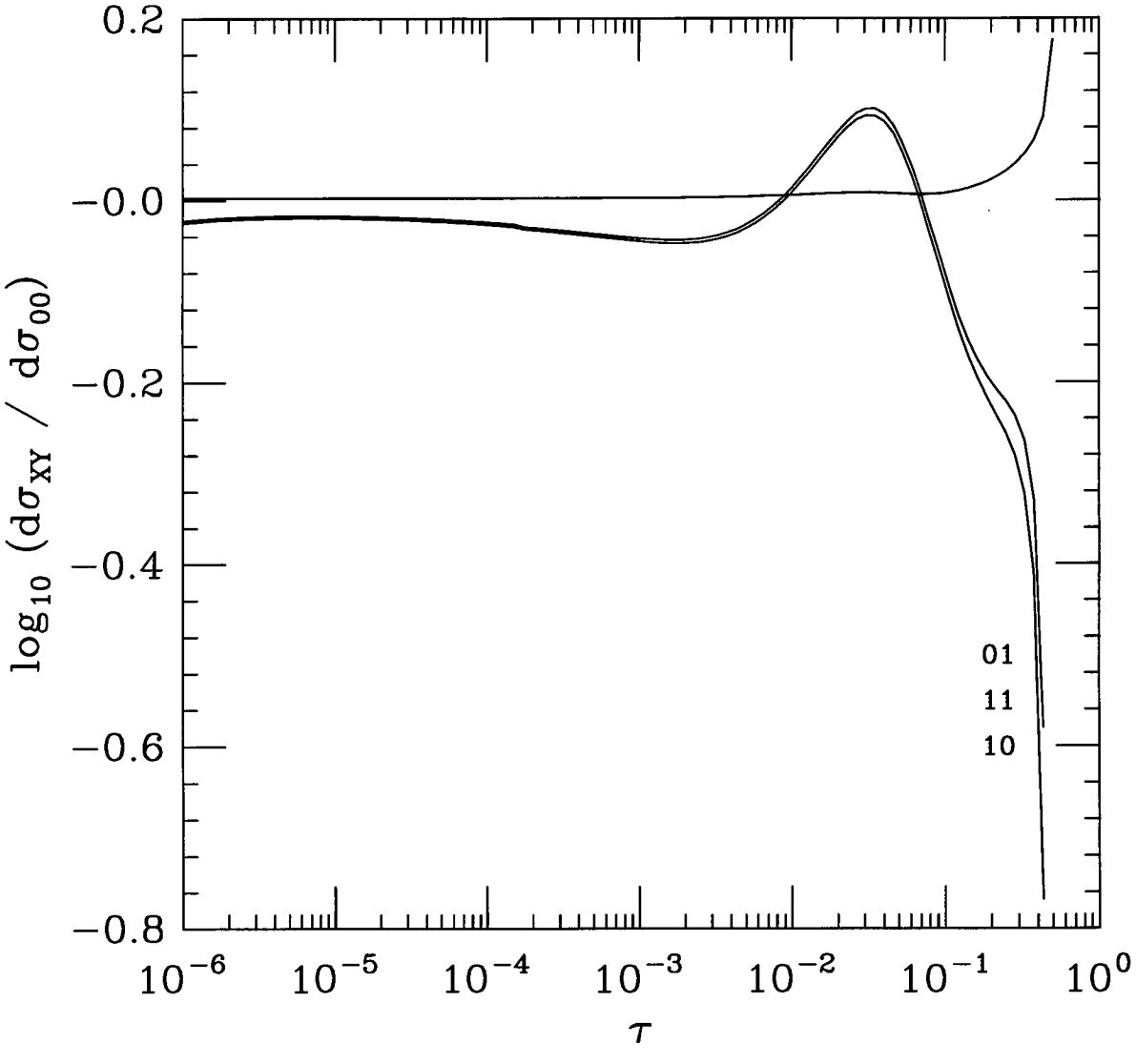

Figure D.7.1: LHC Z production.

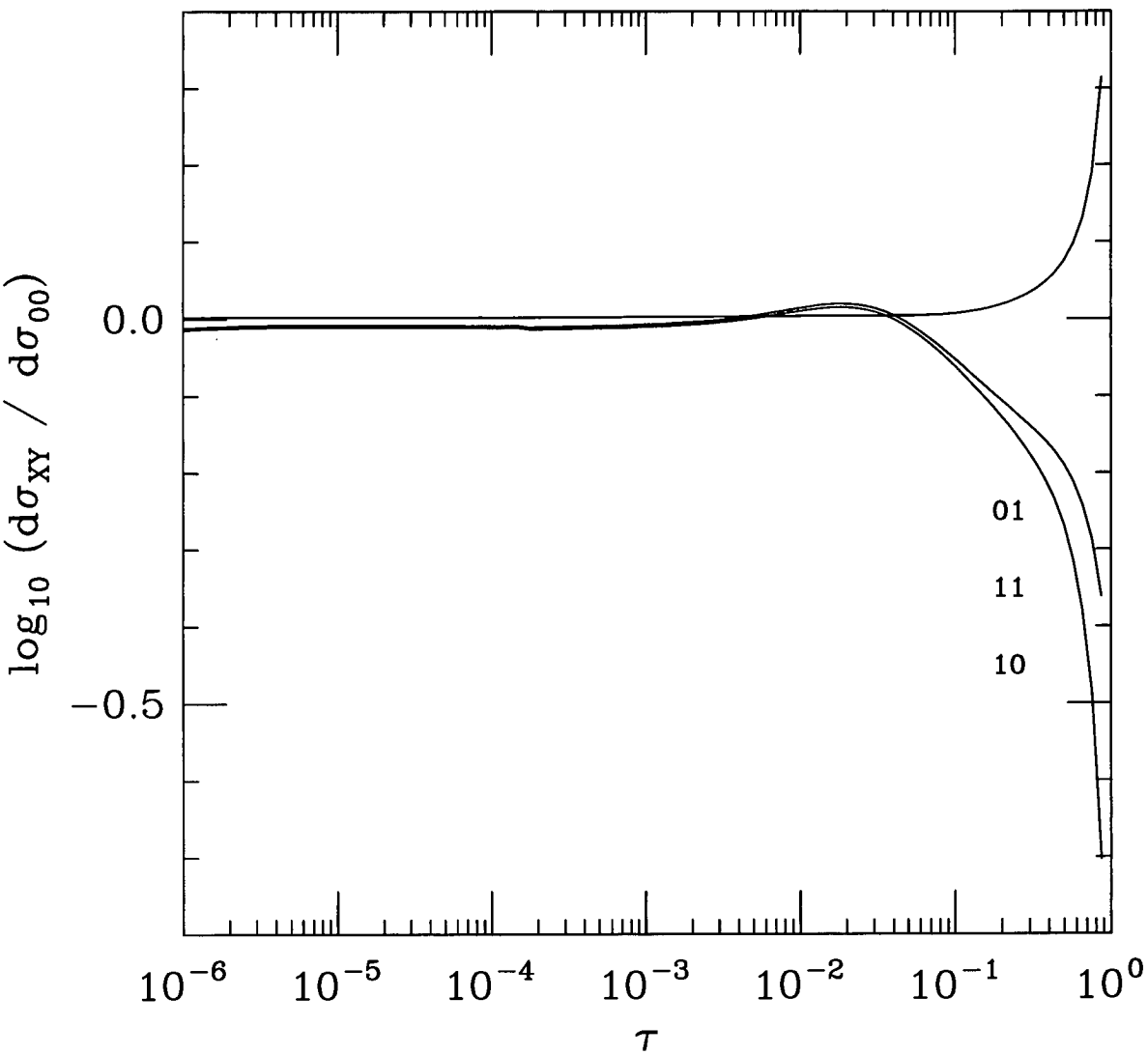

Figure D.7.2: LHC W^+ production.

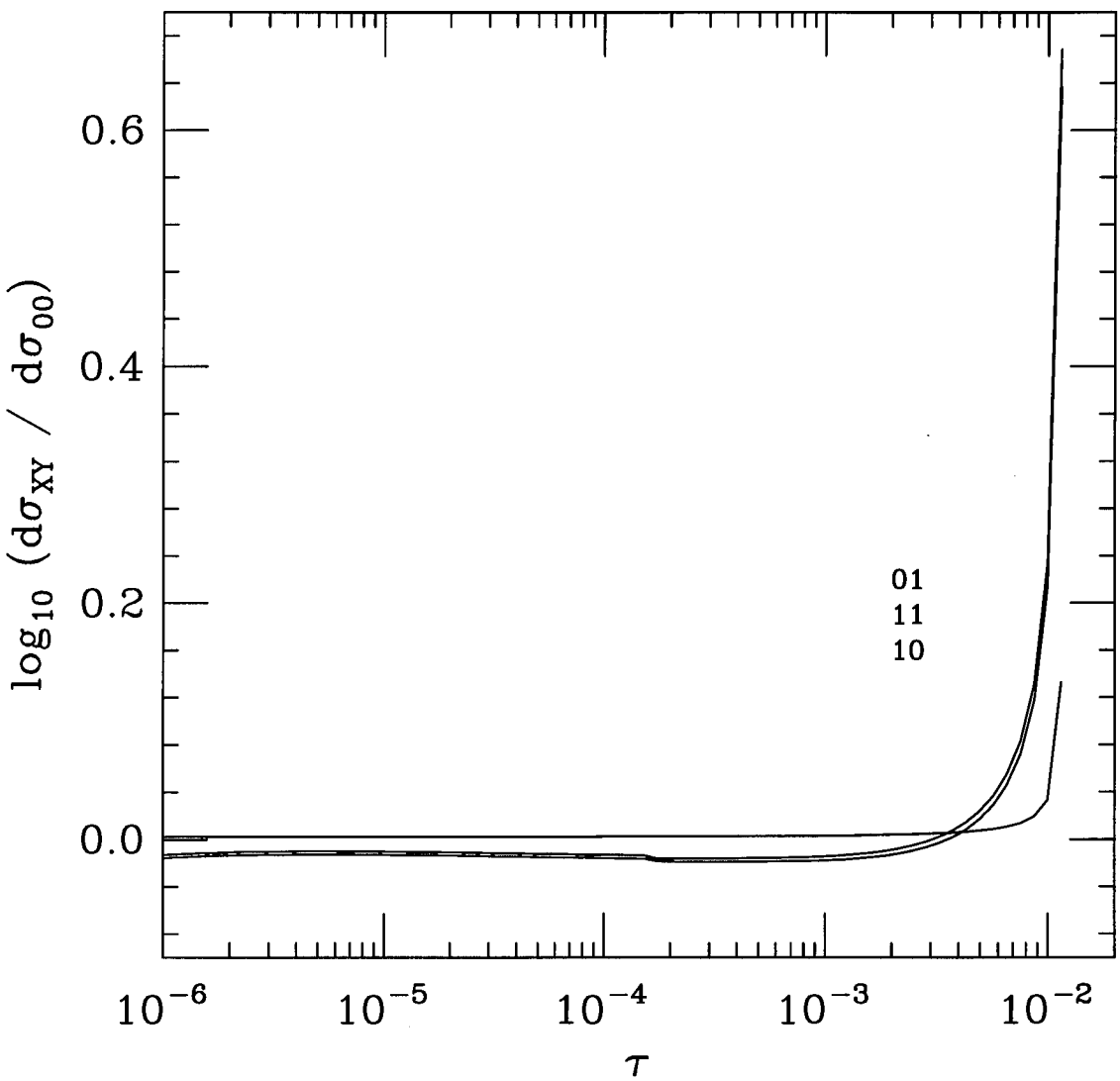

Figure D.7.3: LHC W^- production.

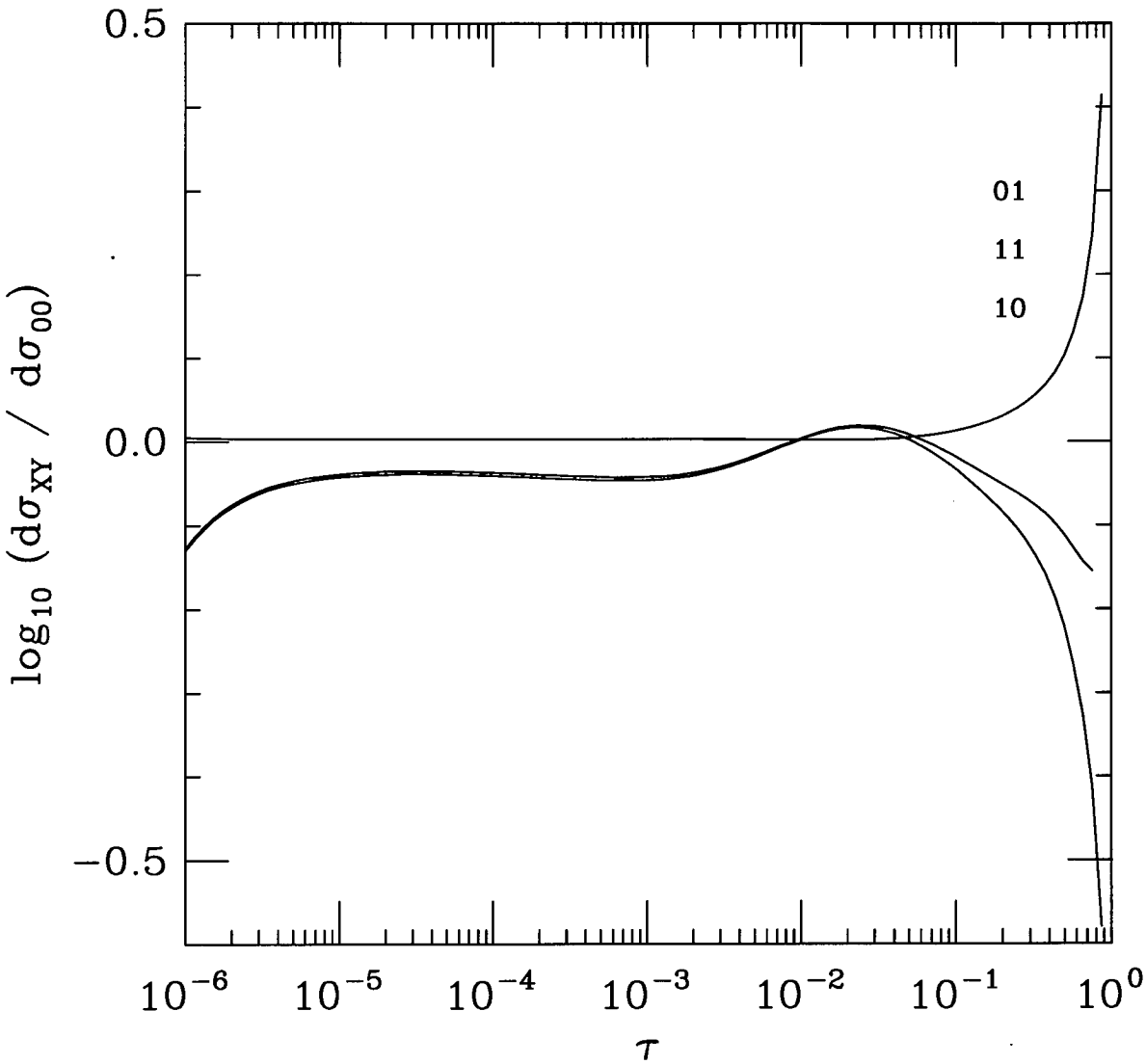

Figure D.7.4: TEVATRON Z production.

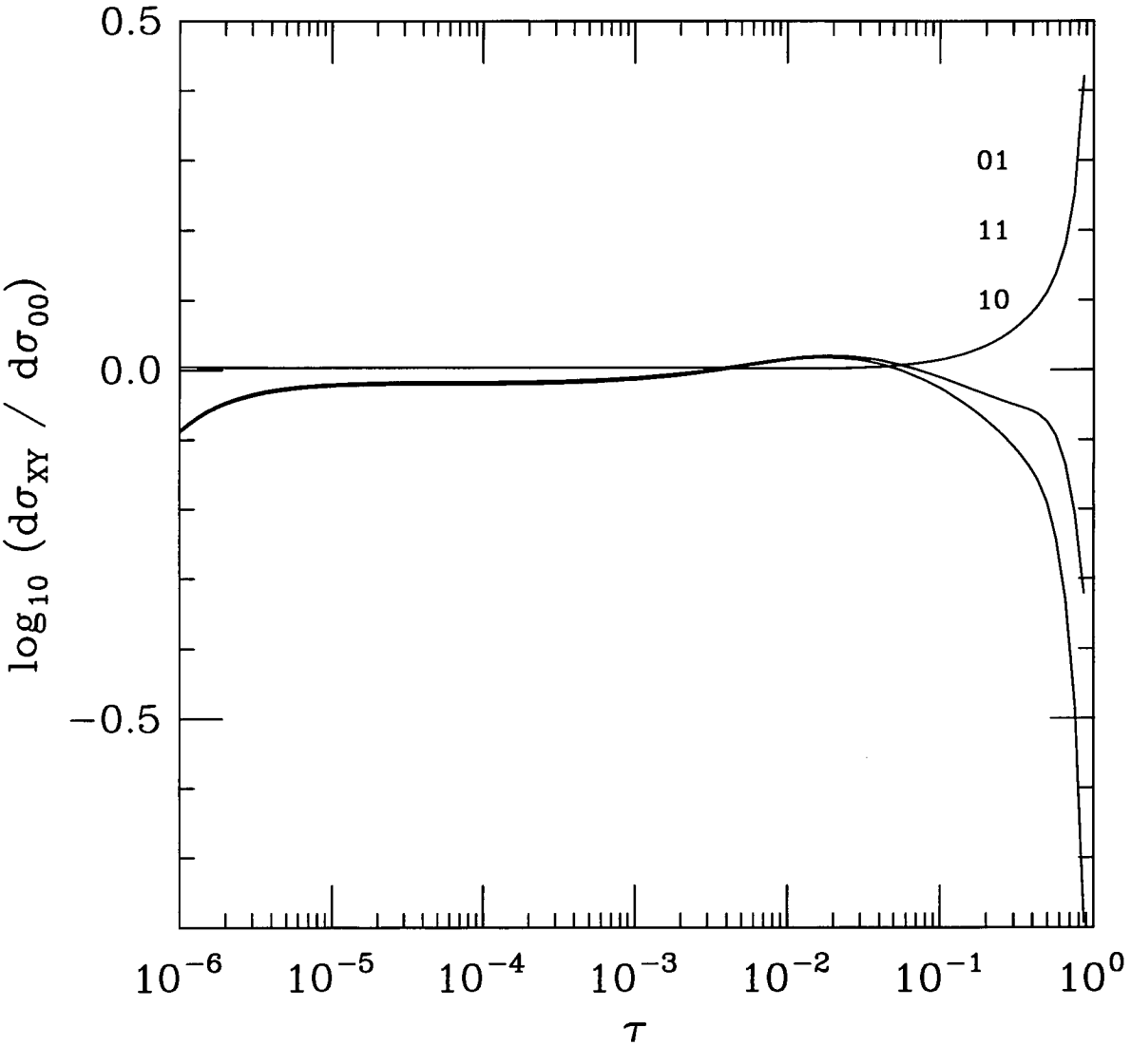

Figure D.7.5: TEVATRON $W^{+/-}$ production.

D.8 Scale Dependence of Drell-Yan cross sections

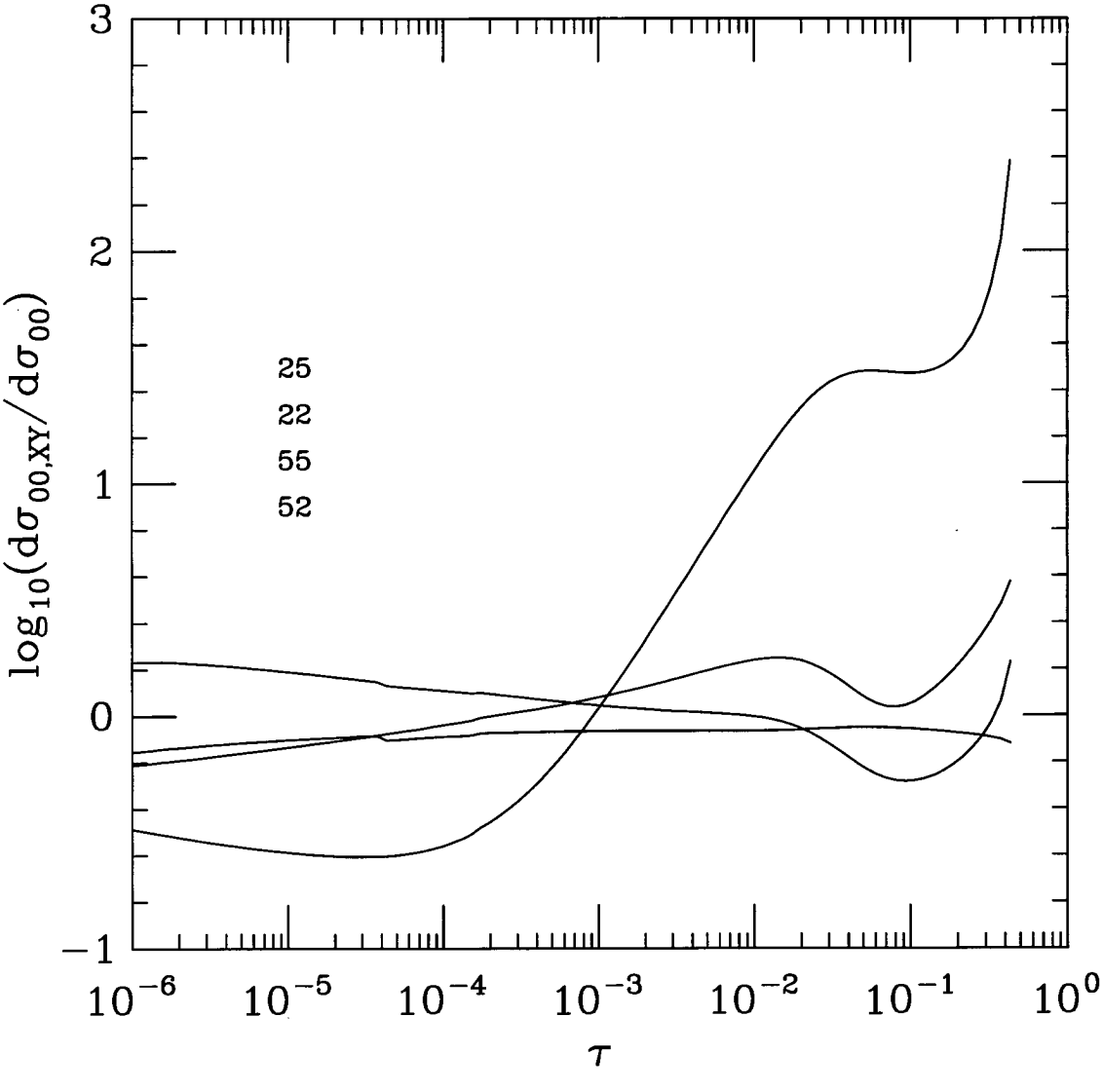

Figure D.8.1: LHC Z production, 00.

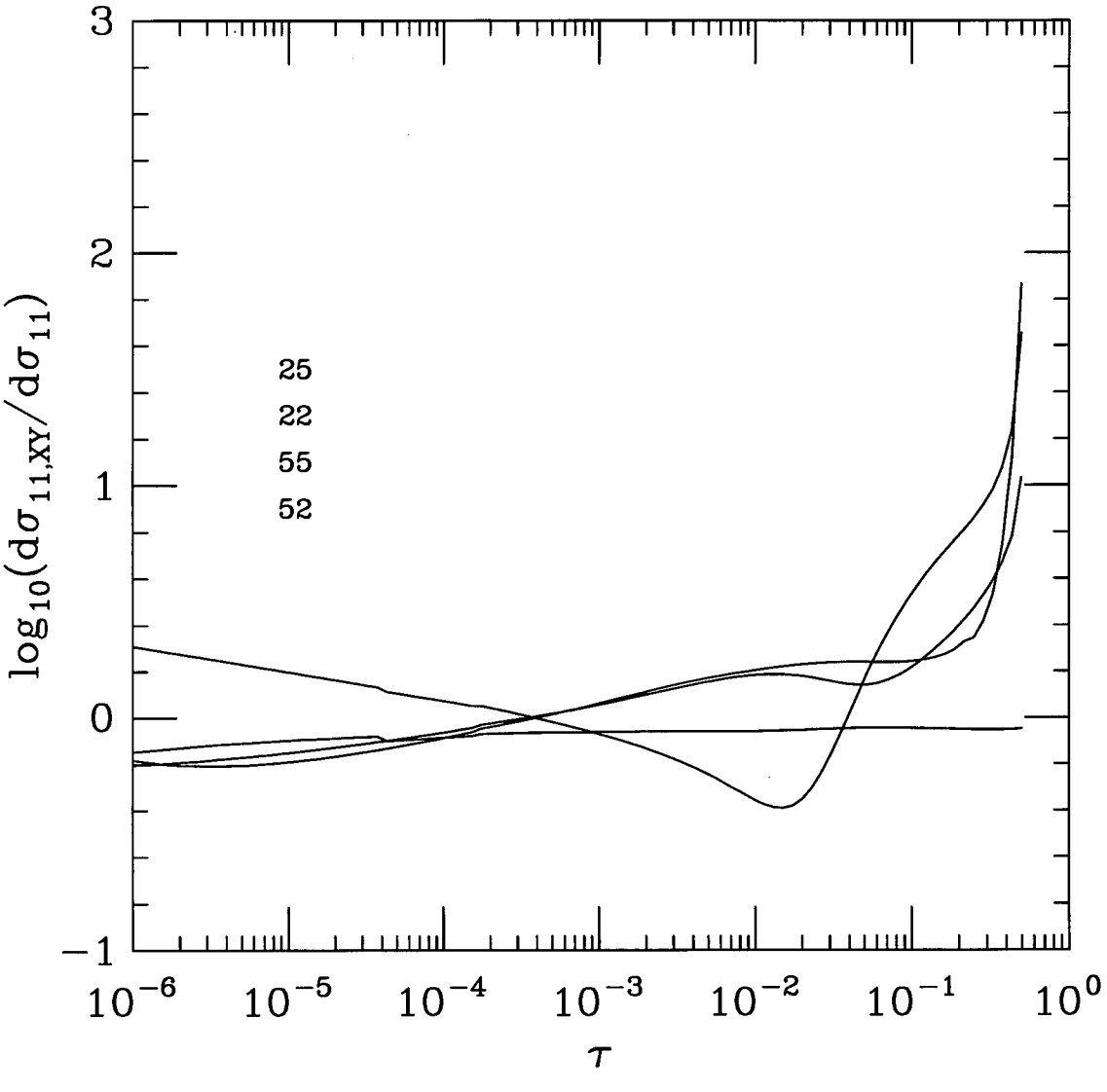

Figure D.8.2: LHC Z production, 11.

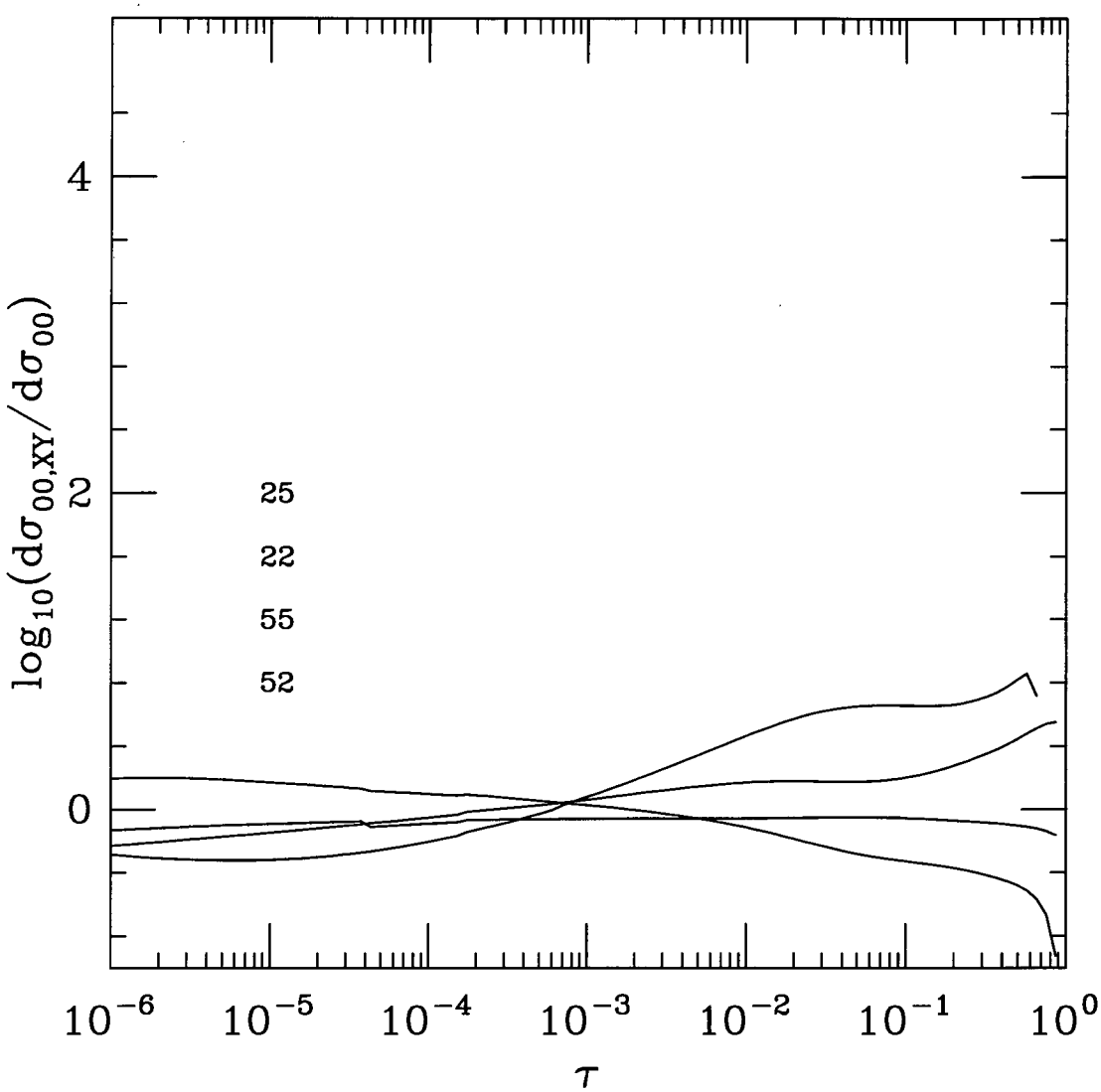

Figure D.8.3: LHC W^+ production, 00.

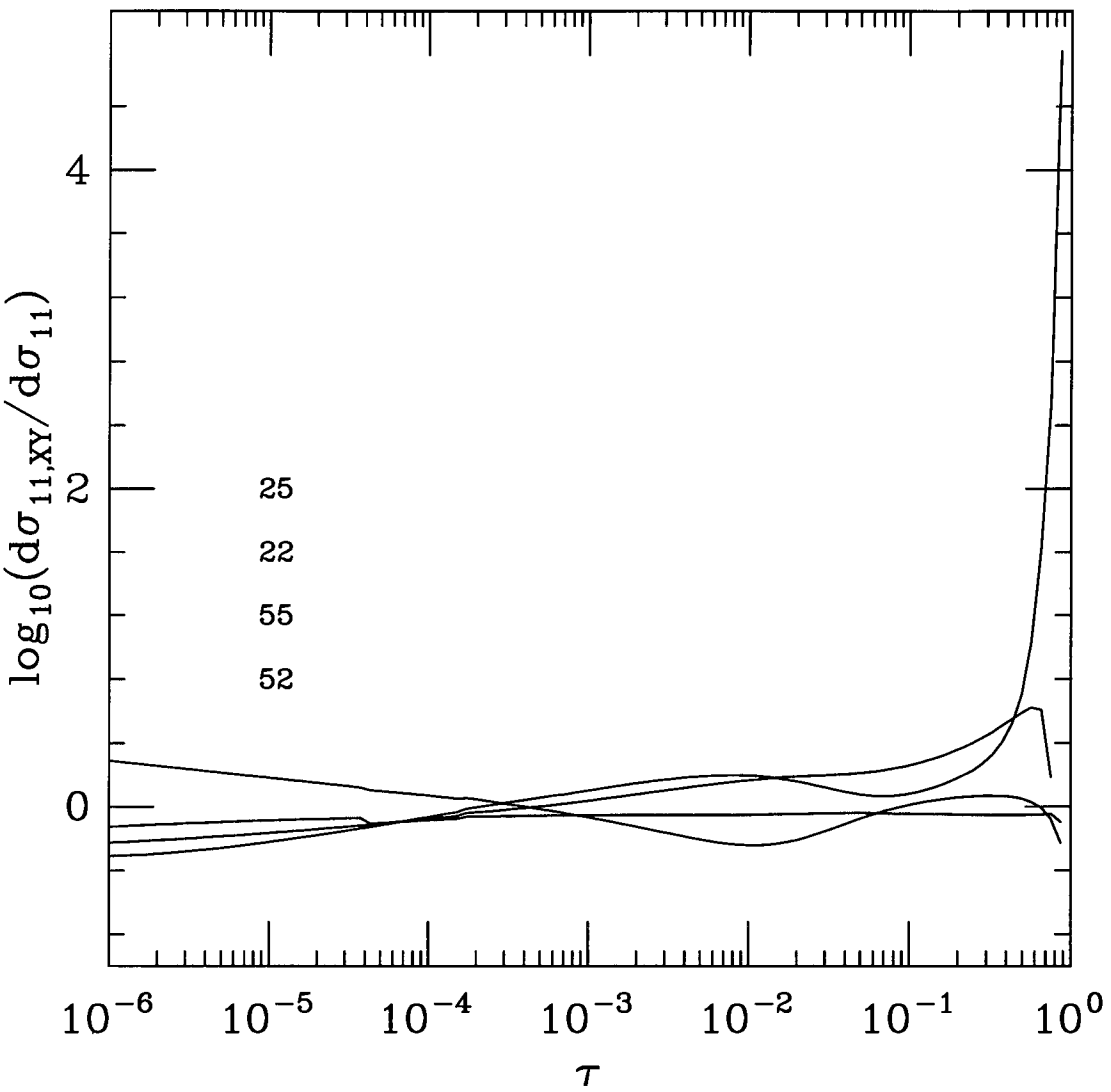

Figure D.8.4: LHC W^+ production, 11.

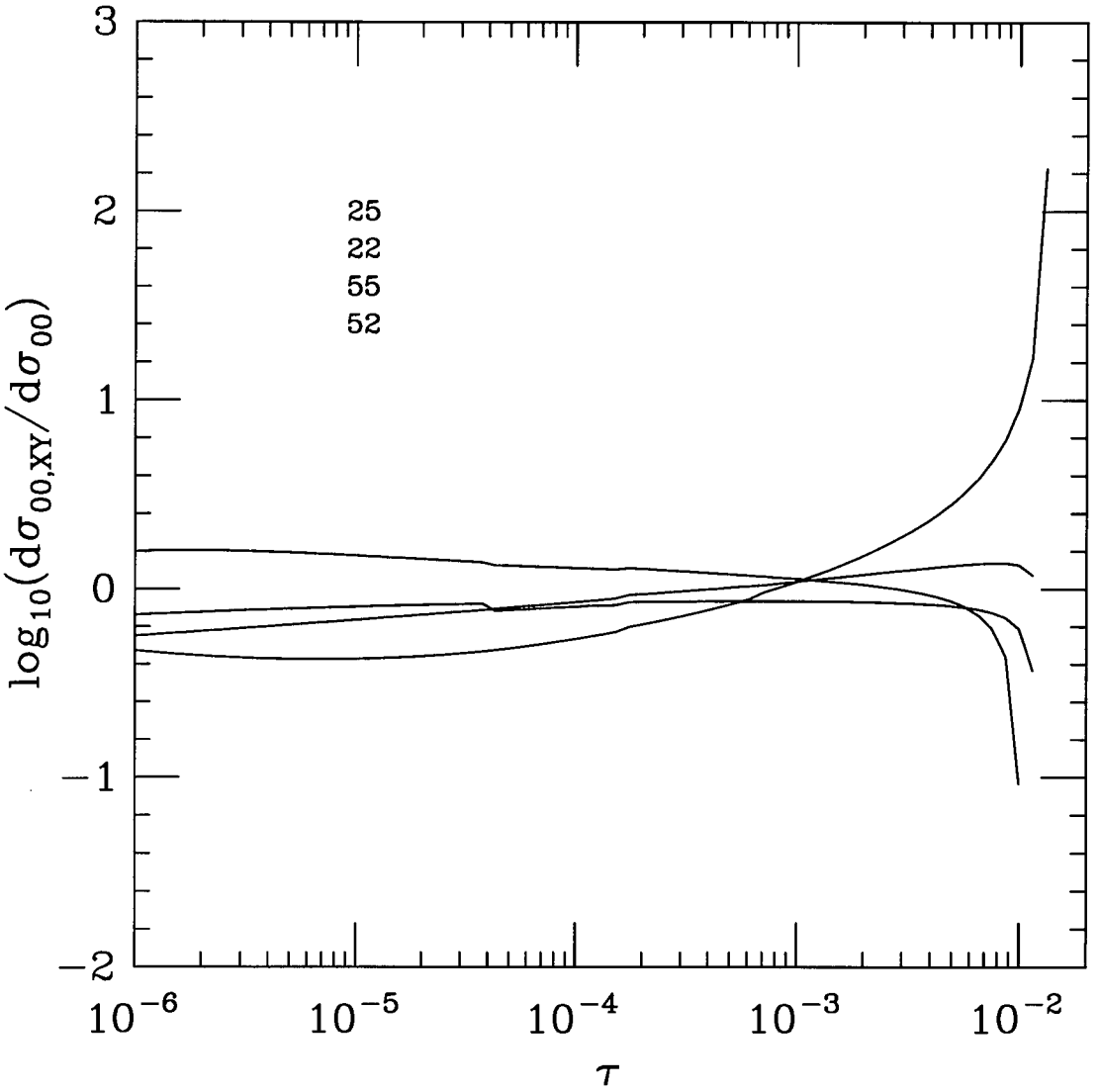

Figure D.8.5: LHC W^- production, 00.

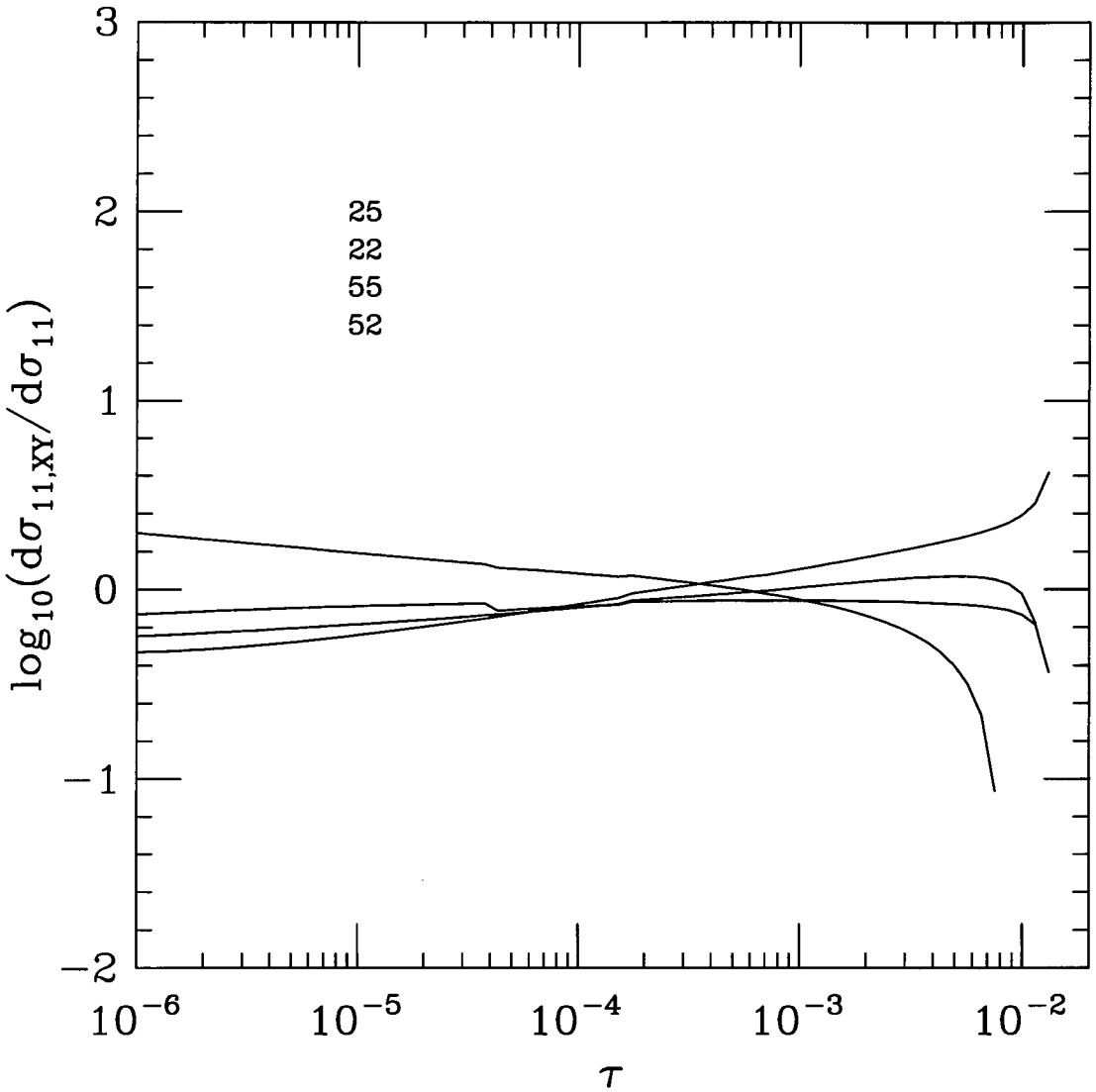

Figure D.8.6: LHC W^- production, 11.

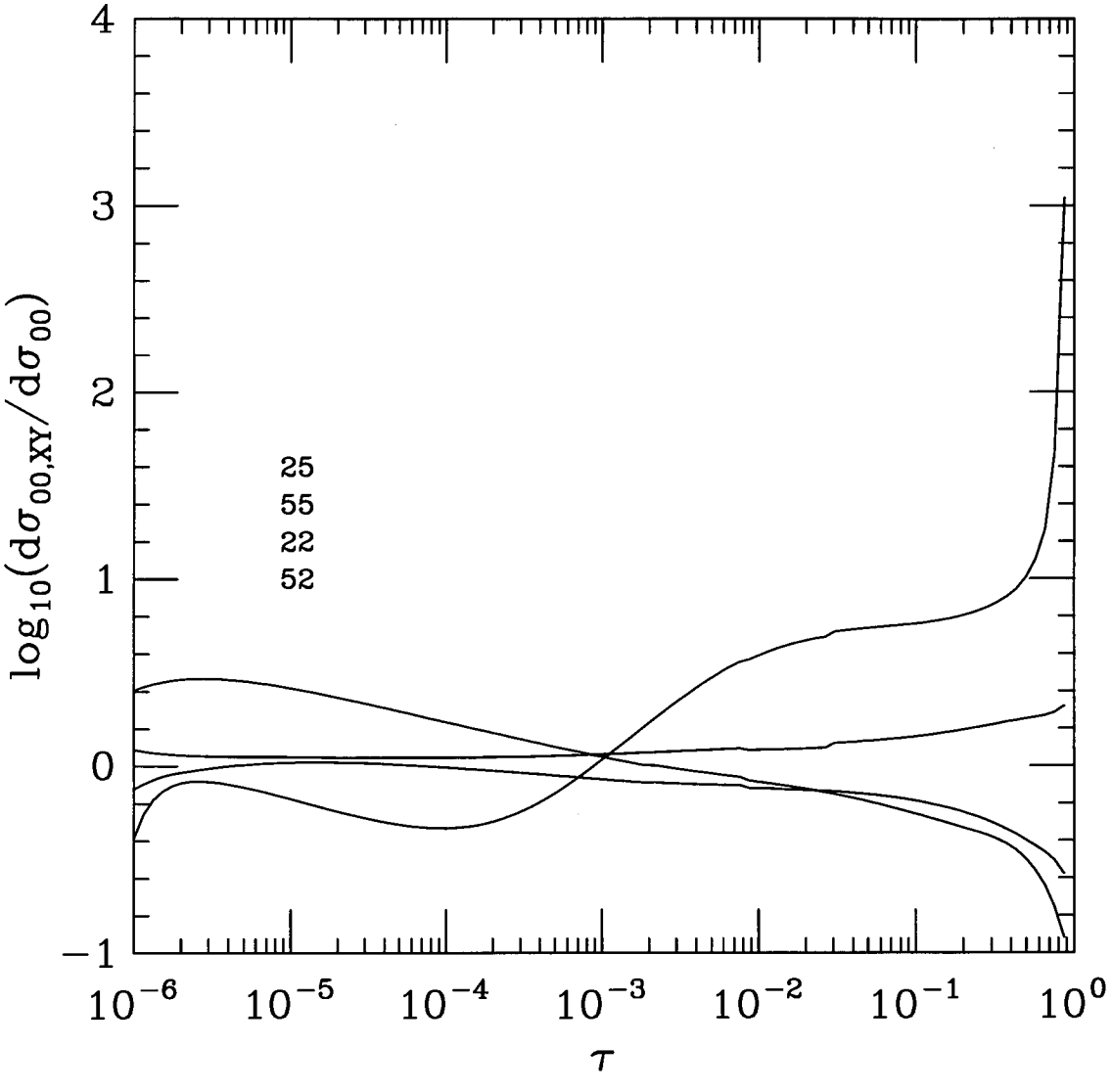

Figure D.8.7: TEVATRON Z production, 00.

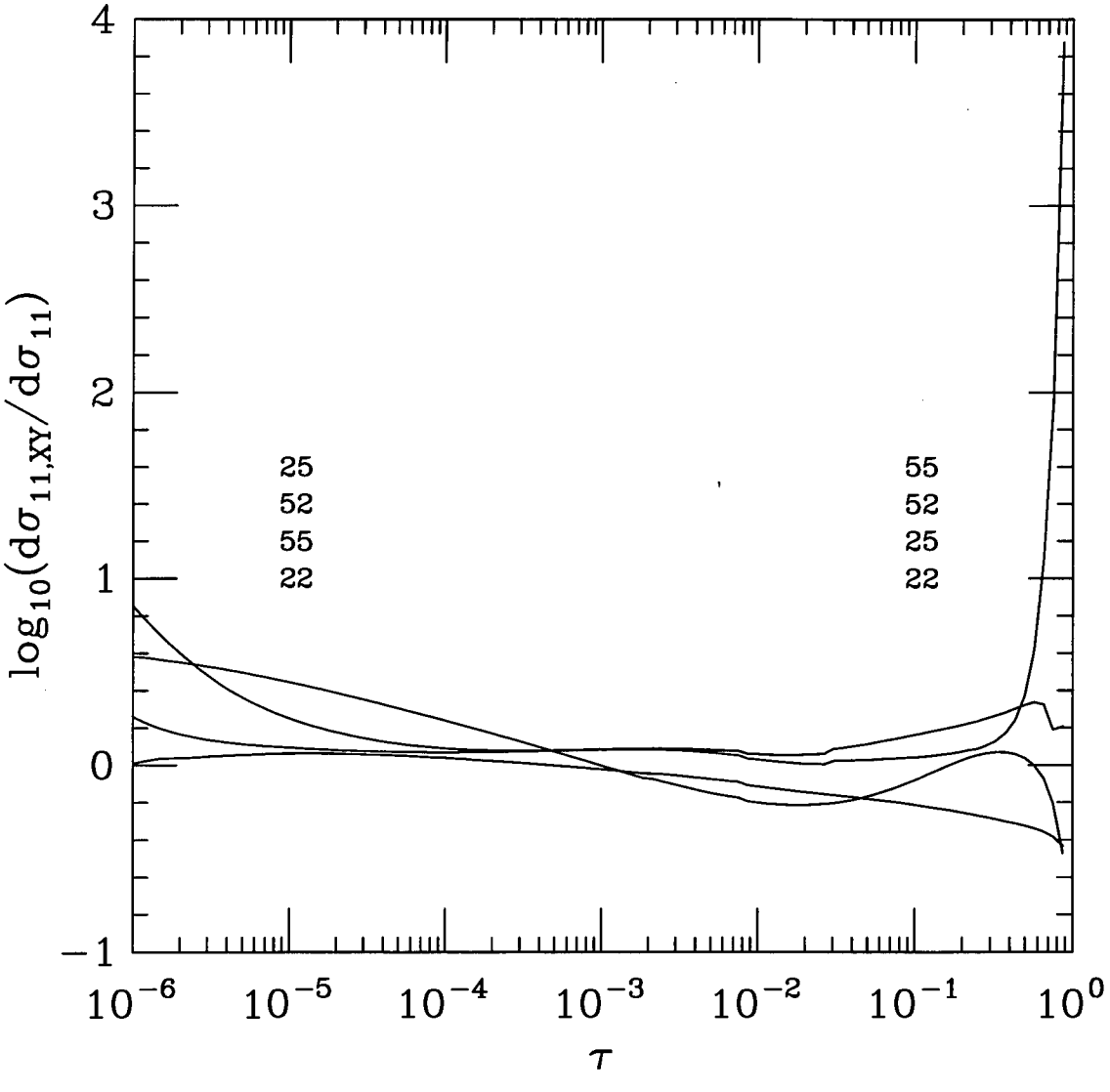

Figure D.8.8: TEVATRON Z production, 11.

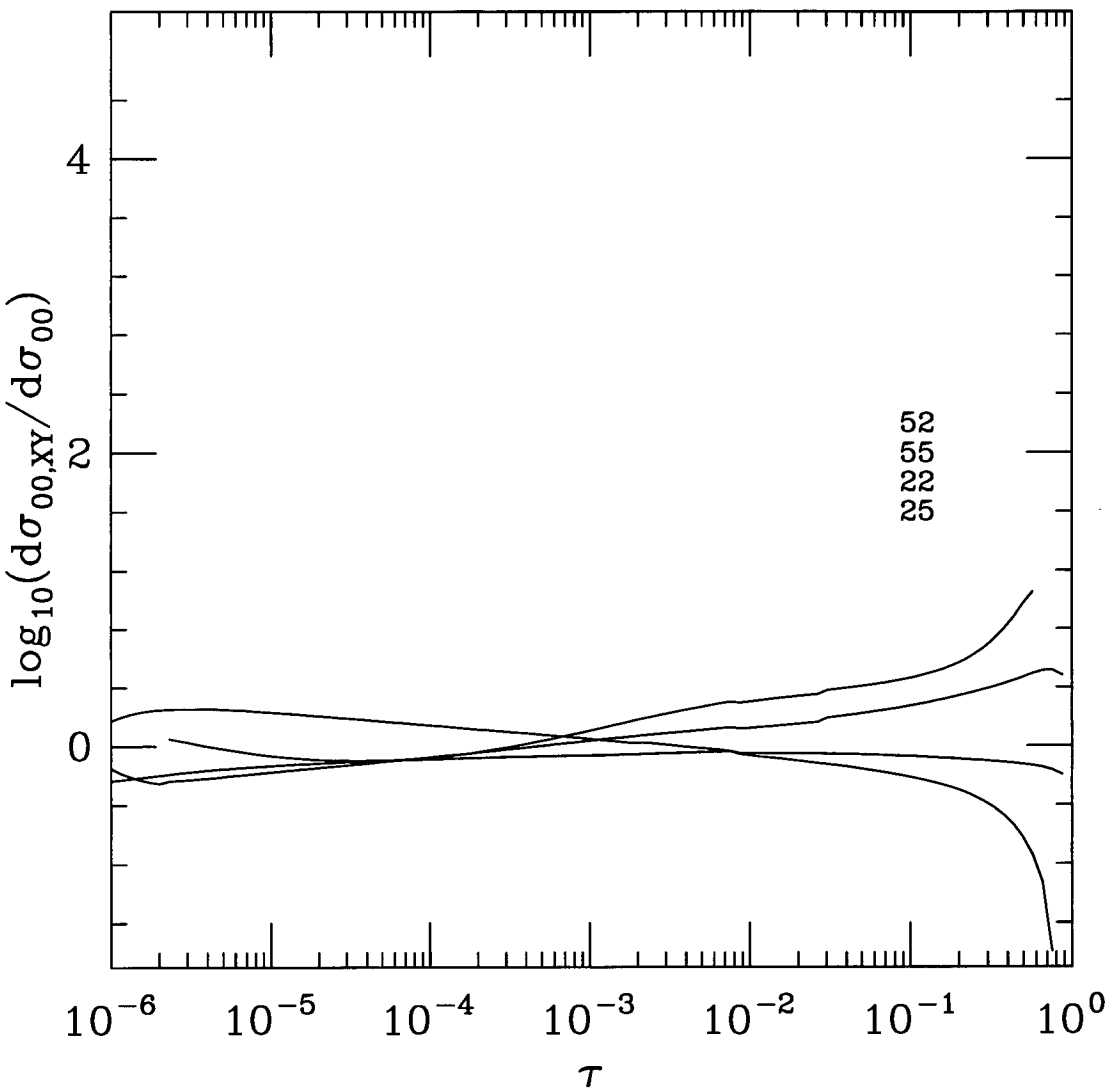

Figure D.8.9: TEVATRON $W^{+/-}$ production, 00.

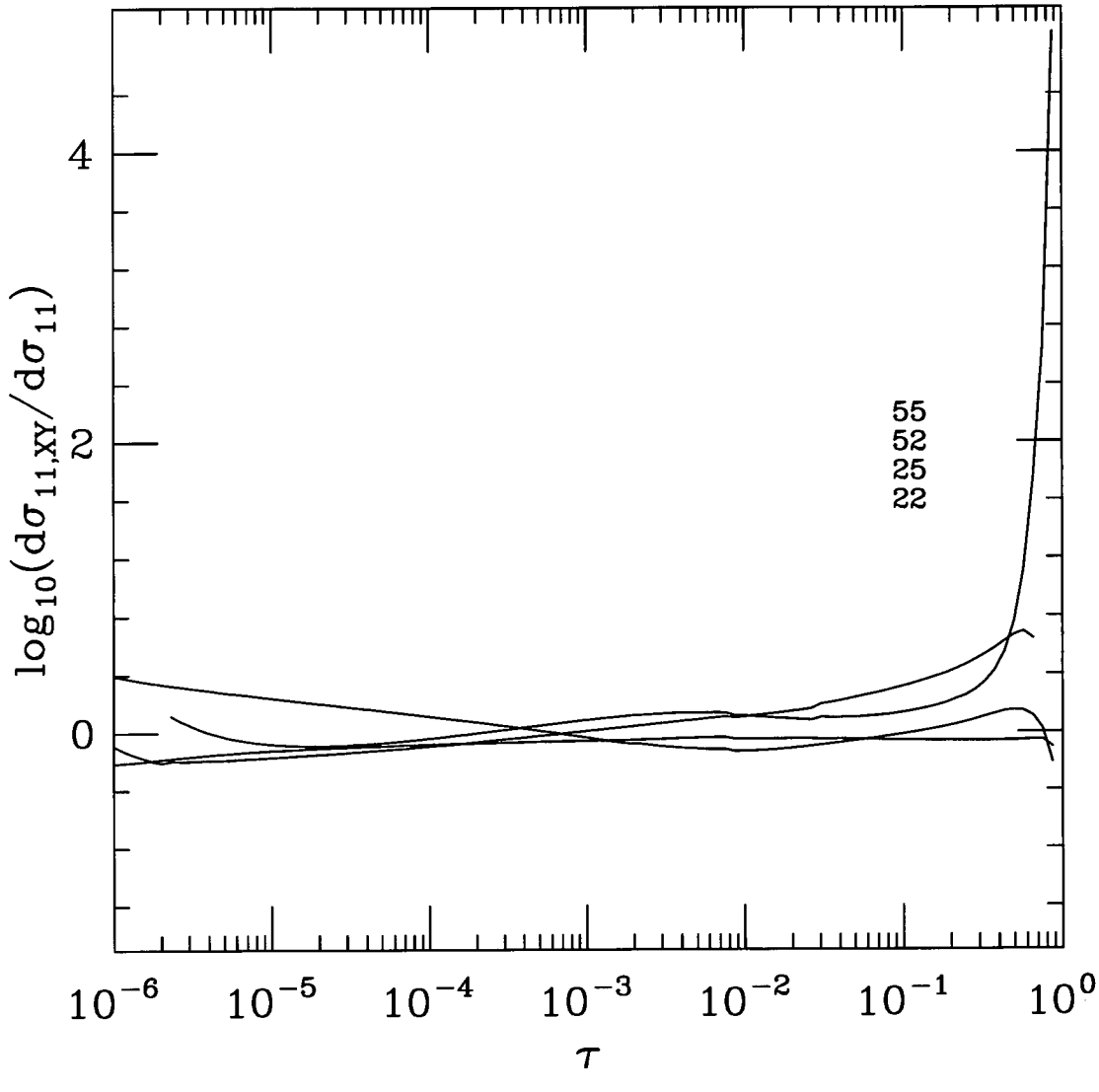

Figure D.8.10: TEVATRON $W^{+/-}$ production, 11.

Appendix E

Results For Heavy Quark Effects

E.1 PDF Parameters

Parameter	Value	Error
a_Σ	-0.34426	0.00666
b_Σ	3.65233	0.01472
c_Σ	3.37988	0.00163
d_Σ	-3.85024	0.00084
A_g	0.95874	0.00064
a_g	-1.81421	0.03084
b_g	0.85991	0.12420
c_g	-9.17496	0.68263
d_g	4.10612	0.41976
A_3	0.09666	0.00776
a_3	2.51712	0.16571
b_3	5.67686	0.11436
c_3	1.91712	0.01974
d_3	-2.61749	0.01449
A_8	2.19467	0.02880
a_8	1.31340	0.01560
b_8	6.50029	0.02421
c_8	1.56814	0.01138
d_8	-2.09897	0.00981
A_H	-1219.26059	663.79682
a_H	3.18084	0.24049
b_H	16.43797	1.37907
c_H	21.03368	2.06494
d_H	-11.47532	1.02891

Table E.1.1: Singlet and non singlet PDF fits in the VFNS for $t_s = 1$. $\widetilde{\chi}^2 = 0.9770$

Parameter	Value	Error
a_Σ	-1.04803	0.01723
b_Σ	3.25096	0.02001
c_Σ	3.28544	0.15206
d_Σ	-1.13238	0.22811
A_g	0.50516	0.01476
a_g	-1.77715	0.49427
b_g	5.78632	5.37221
c_g	664.48229	117.30115
d_g	-317.26899	55.44220
A_3	0.09667	0.00752
a_3	2.39223	0.13591
b_3	5.65564	0.09655
c_3	1.91665	0.02072
d_3	-2.60483	0.01546
A_8	0.51090	0.04620
a_8	0.58765	0.06559
b_8	6.80771	0.10321
c_8	3.02989	0.08799
d_8	-2.83414	0.05956
A_H	-0.47306	0.05864
a_H	0.17260	0.12608
b_H	-0.81177	0.10088
c_H	-3.00161	0.26524
d_H	0.41491	0.21960

Table E.1.2: Singlet and non singlet PDF fits in the VFNS for $t_s = 4$. $\widetilde{\chi}^2 = 0.9460$

Parameter	Value	Error
a_Σ	-1.05763	0.00899
b_Σ	2.96230	0.01923
c_Σ	5.33111	0.02038
d_Σ	-4.81244	0.00789
A_g	0.92612	0.00179
a_g	-1.99948	0.00052
b_g	6.17469	1.69396
c_g	5491.89478	3371.12621
d_g	-1113.15424	659.87354
A_3	0.09456	0.00548
a_3	2.63269	0.09092
b_3	5.84416	0.07384
c_3	1.93992	0.01399
d_3	-2.64782	0.01061
A_8	2.24438	0.03049
a_8	-0.42339	0.01107
b_8	4.68554	0.02209
c_8	1.61206	0.05646
d_8	-0.66356	0.07507
A_H	-519.29587	93.62783
a_H	1.43566	0.07214
b_H	12.55534	0.51849
c_H	5.75068	0.07887
d_H	-4.67126	0.03414

Table E.1.3: Singlet and non singlet PDF fits in the VFNS for $t_s = \frac{1}{4}$. $\tilde{\chi}^2 = 1.0500$

Parameter	Value	Error
a_Σ	-1.50169	0.01397
b_Σ	3.34071	0.01929
c_Σ	17.47728	1.16925
d_Σ	-5.39481	0.13996
A_g	0.56728	0.01582
a_g	-1.62455	0.09672
b_g	-0.34287	0.07739
c_g	2.13101	0.03146
d_g	-3.01492	0.02495
A_3	0.09218	0.00595
a_3	2.67337	0.10165
b_3	6.06066	0.08310
c_3	1.98221	0.01702
d_3	-2.66643	0.01266
A_8	0.92622	0.04180
a_8	-0.25753	0.03467
b_8	5.66745	0.07238
c_8	3.12663	0.05170
d_8	-2.86397	0.03444
A_H	-0.10233	0.00930
a_H	-0.53547	0.11706
b_H	-2.44544	0.05765
c_H	-2.05308	0.12040
d_H	-0.22420	0.10839

Table E.1.4: Singlet and non singlet PDF fits in the ZM-VFNS for $t_s = 1$. $\tilde{\chi}^2 = 0.9340$

Parameter	Value	Error
a_Σ	-1.06064	0.00156
b_Σ	3.43332	0.00312
c_Σ	4.41169	0.02365
d_Σ	-0.80459	0.06162
A_g	0.36615	0.00401
a_g	-1.78606	0.09287
b_g	24.91681	2.33782
c_g	486.74372	402.19898
d_g	45.44846	74.77680
A_3	0.09692	0.00158
a_3	2.40547	0.01521
b_3	5.68250	0.01906
c_3	1.92837	0.00420
d_3	-2.61505	0.00369
A_8	0.00065	0.00016
a_8	2.47067	0.00546
b_8	5.11371	0.00686
c_8	-3.19584	0.00181
d_8	0.55853	0.00120
A_H	-0.11413	0.00253
a_H	-0.64428	0.01451
b_H	-1.51602	0.02953
c_H	-3.65340	0.06628
d_H	0.68376	0.05073

Table E.1.5: Singlet and non singlet PDF fits in the ZM-VFNS for $t_s = 4$. $\tilde{\chi}^2 = 0.9480$

Parameter	Value	Error
a_Σ	-0.28704	0.01239
b_Σ	3.26735	0.01516
c_Σ	1.19654	0.00796
d_Σ	-1.87538	0.00763
A_g	0.46226	0.00791
a_g	-1.31776	0.11837
b_g	29.56083	4.18661
c_g	41.15242	1.91856
d_g	-14.18703	0.29842
A_3	0.09352	0.00614
a_3	2.67049	0.10749
b_3	5.84557	0.08175
c_3	1.92392	0.01476
d_3	-2.63970	0.01122
A_8	0.14890	0.00799
a_8	5.64408	0.12139
b_8	10.79812	0.13293
c_8	2.10106	0.01099
d_8	-2.80245	0.00780
A_H	-28.40427	5.29482
a_H	2.67476	0.09681
b_H	13.73023	0.42460
c_H	193.36056	16.51461
d_H	-54.92093	7.40479

Table E.1.6: Singlet and non singlet PDF fits in the ZM-VFNS for $t_s = \frac{1}{4}$. $\tilde{\chi}^2 = 1.0440$

E.2 PDF Plots

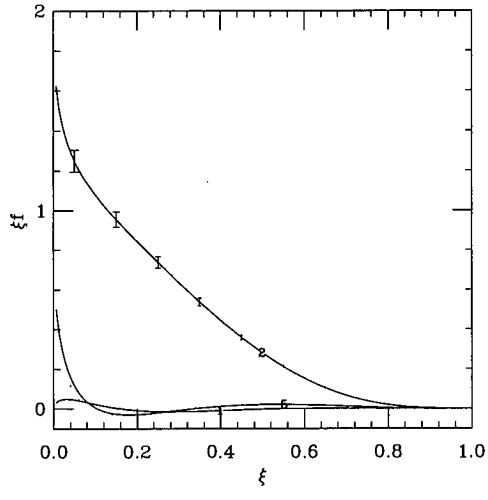

Figure E.2.1: Singlet quark fit in the VFNS.

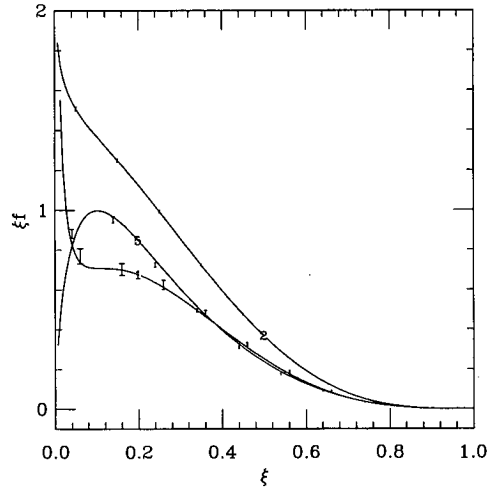

Figure E.2.2: Singlet quark fit in the ZM-VFNS.

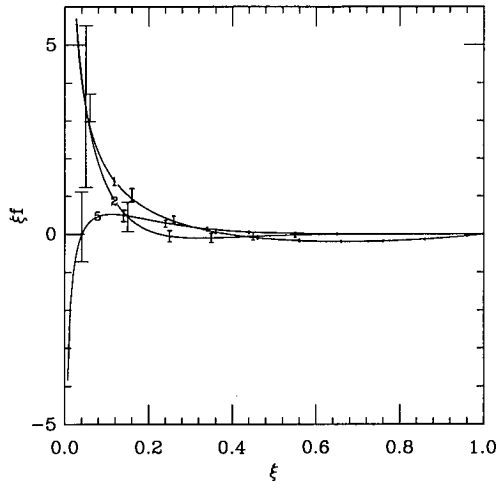

Figure E.2.3: Gluon fit in the VFNS.

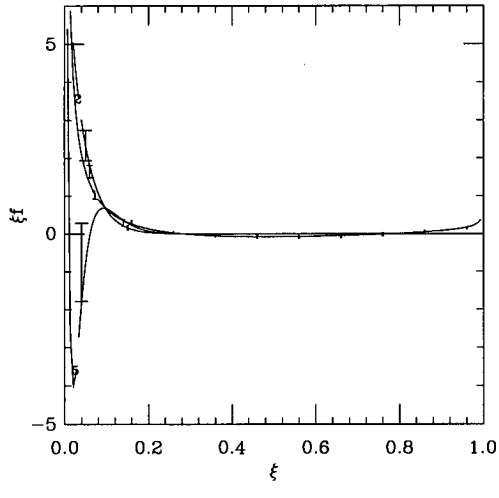

Figure E.2.4: Gluon fit in the ZM-VFNS.

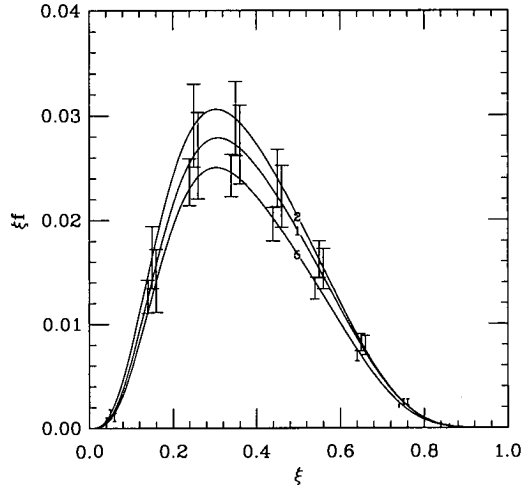

Figure E.2.5: T_3 fit in the VFNS.

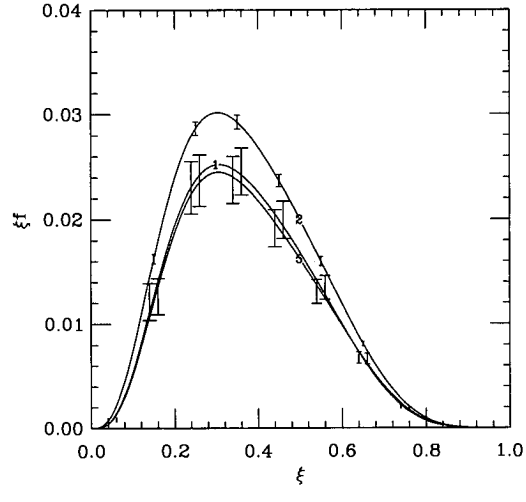

Figure E.2.6: T_3 fit in the ZM-VFNS.

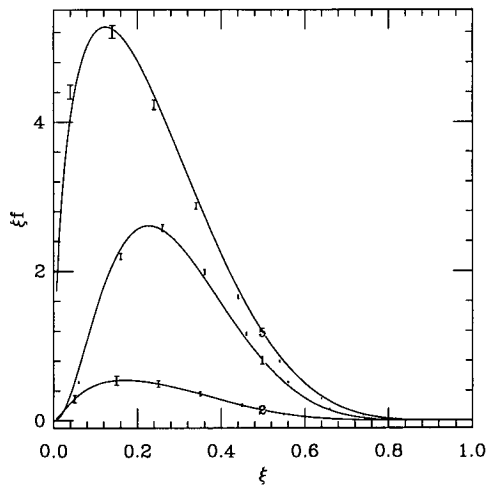

Figure E.2.7: T_8 fit in the VFNS.

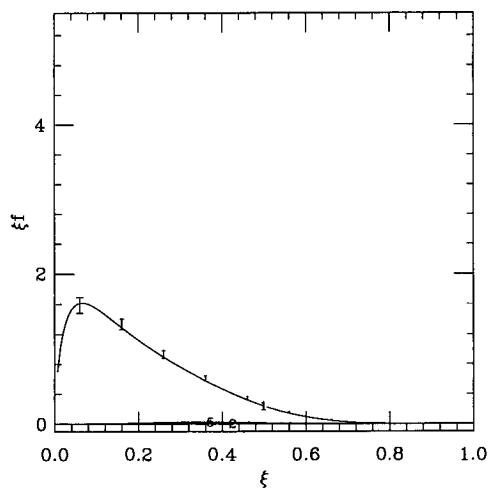

Figure E.2.8: T_8 fit in the ZM-VFNS.

E.3 Higher Twist Plots

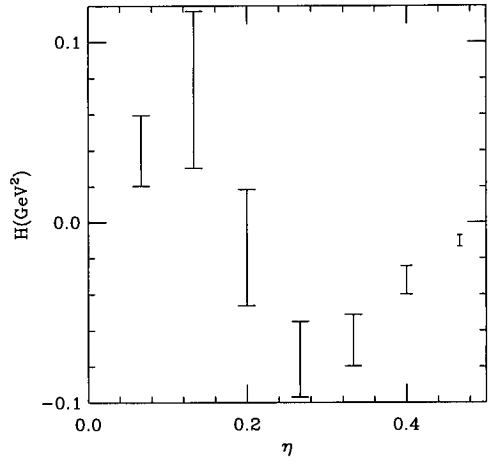

Figure E.3.1: Higher twist fit in the VFNS with $t_s = 1$.

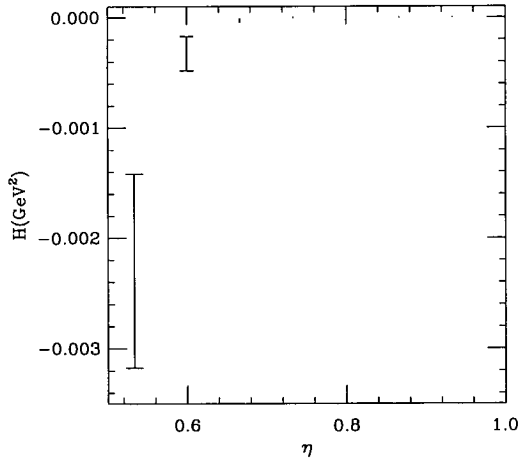

Figure E.3.2: Higher twist fit in the VFNS with $t_s = 1$.

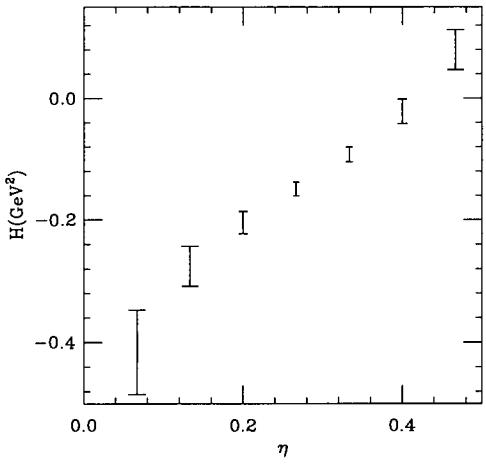

Figure E.3.3: Higher twist fit in the ZM-VFNS with $t_s = 1$.

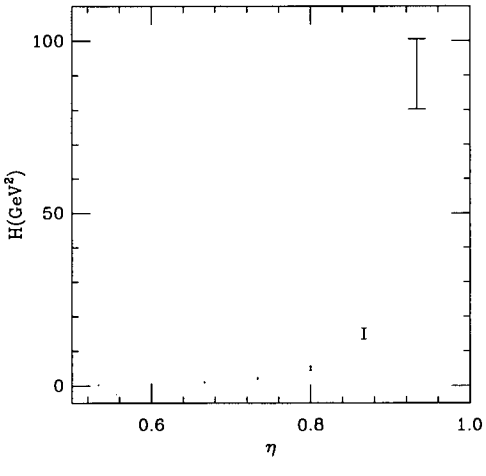

Figure E.3.4: Higher twist fit in the ZM-VFNS with $t_s = 1$.

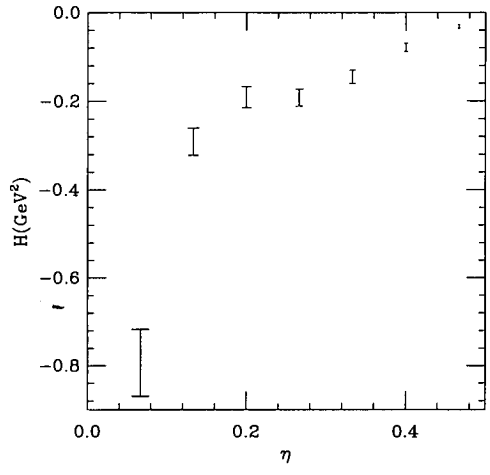

Figure E.3.5: Higher twist fit in the VFNS with $t_s = \frac{1}{4}$.

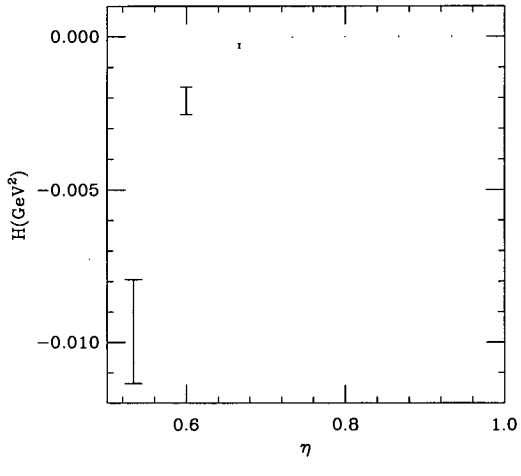

Figure E.3.6: Higher twist fit in the VFNS with $t_s = \frac{1}{4}$.

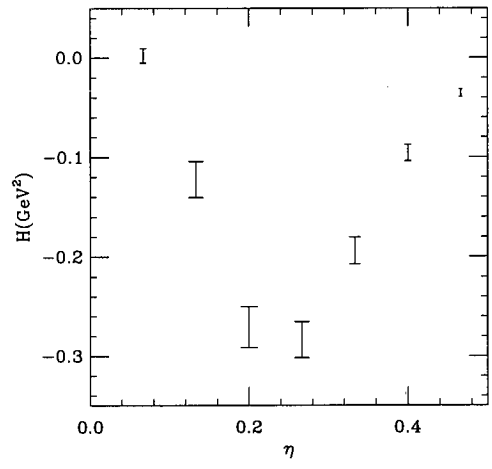

Figure E.3.7: Higher twist fit in the ZM-VFNS with $t_s = \frac{1}{4}$.

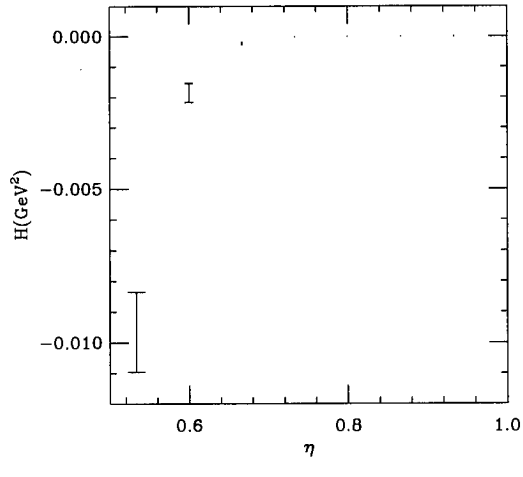

Figure E.3.8: Higher twist fit in the ZM-VFNS with $t_s = \frac{1}{4}$.

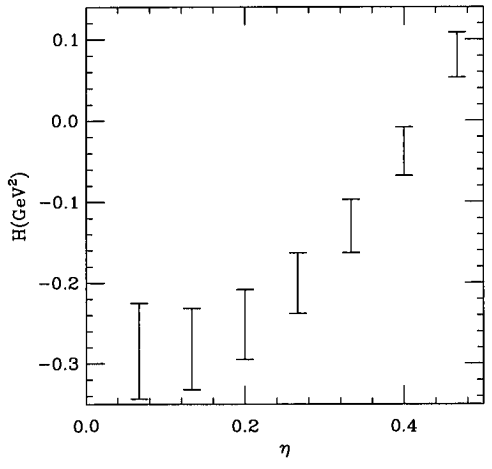

Figure E.3.9: Higher twist fit in the VFNS with $t_s = 4$.

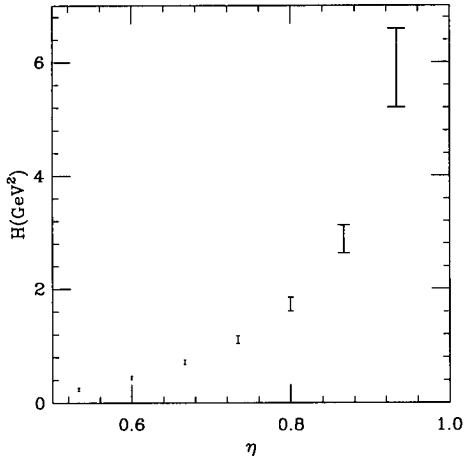

Figure E.3.10: Higher twist fit in the VFNS with $t_s = 4$.

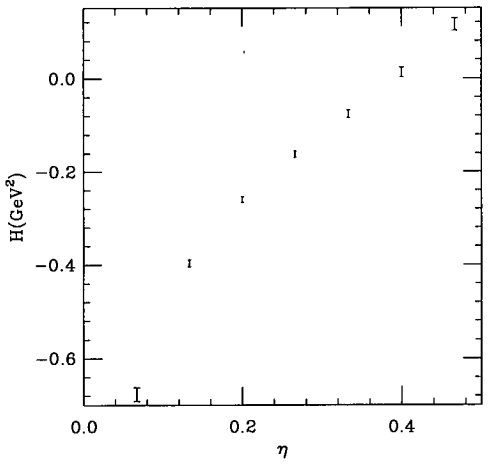

Figure E.3.11: Higher twist fit in the ZM-VFNS with $t_s = 4$.

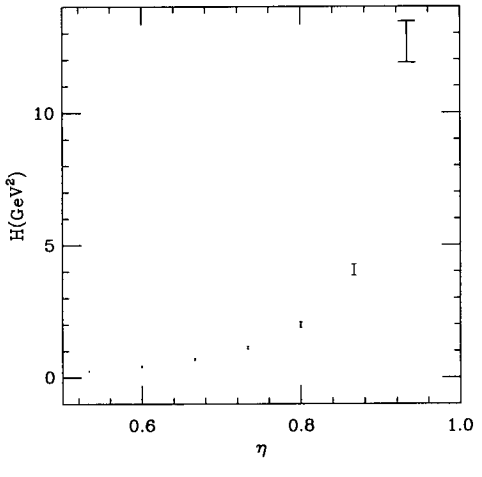

Figure E.3.12: Higher twist fit in the ZM-VFNS with $t_s = 4$.

E.4 F_2 At Different Threshold Scales

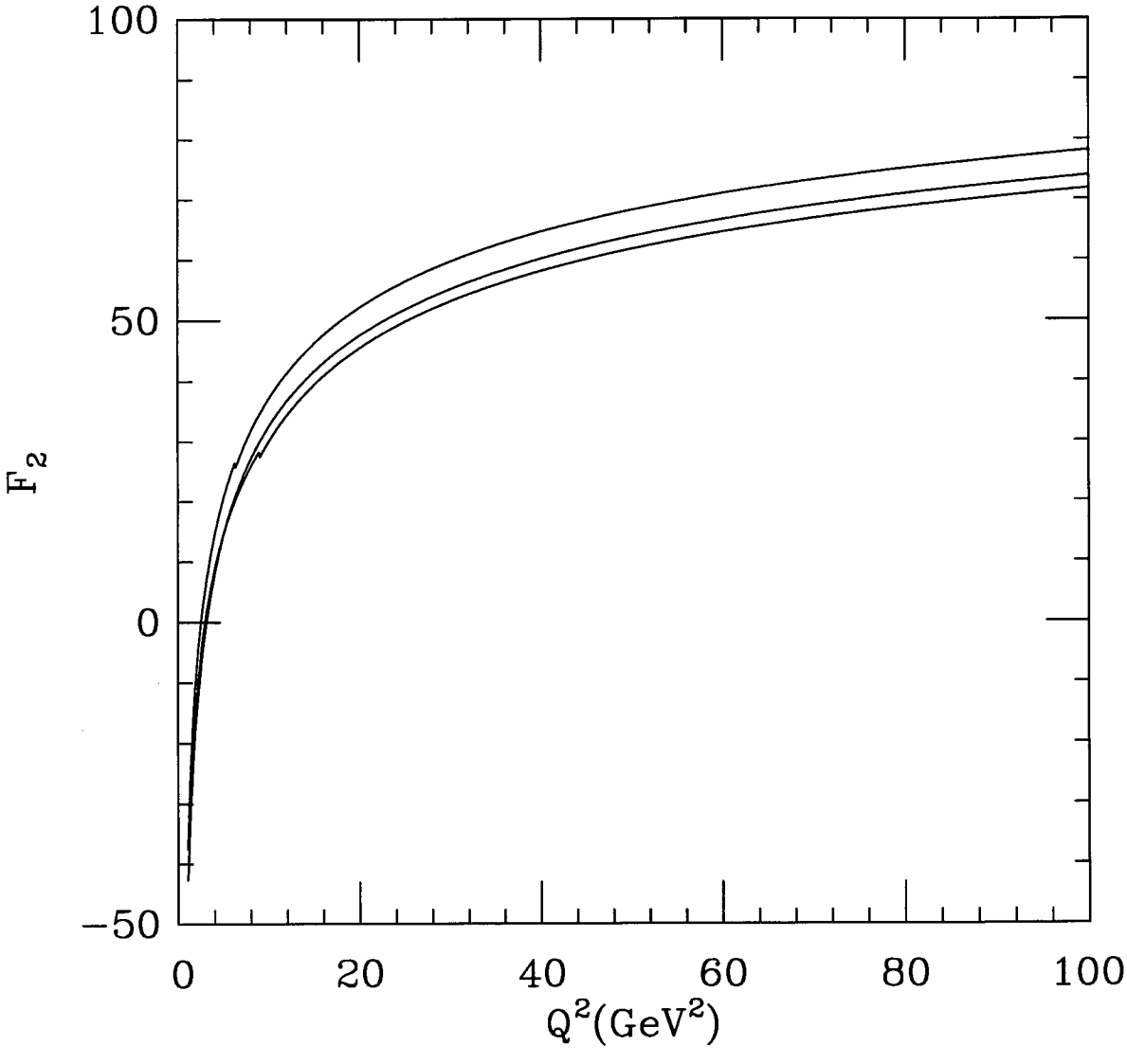

Figure E.4.1: F_2 cross section in the VFNS for $\eta = 0.01$.

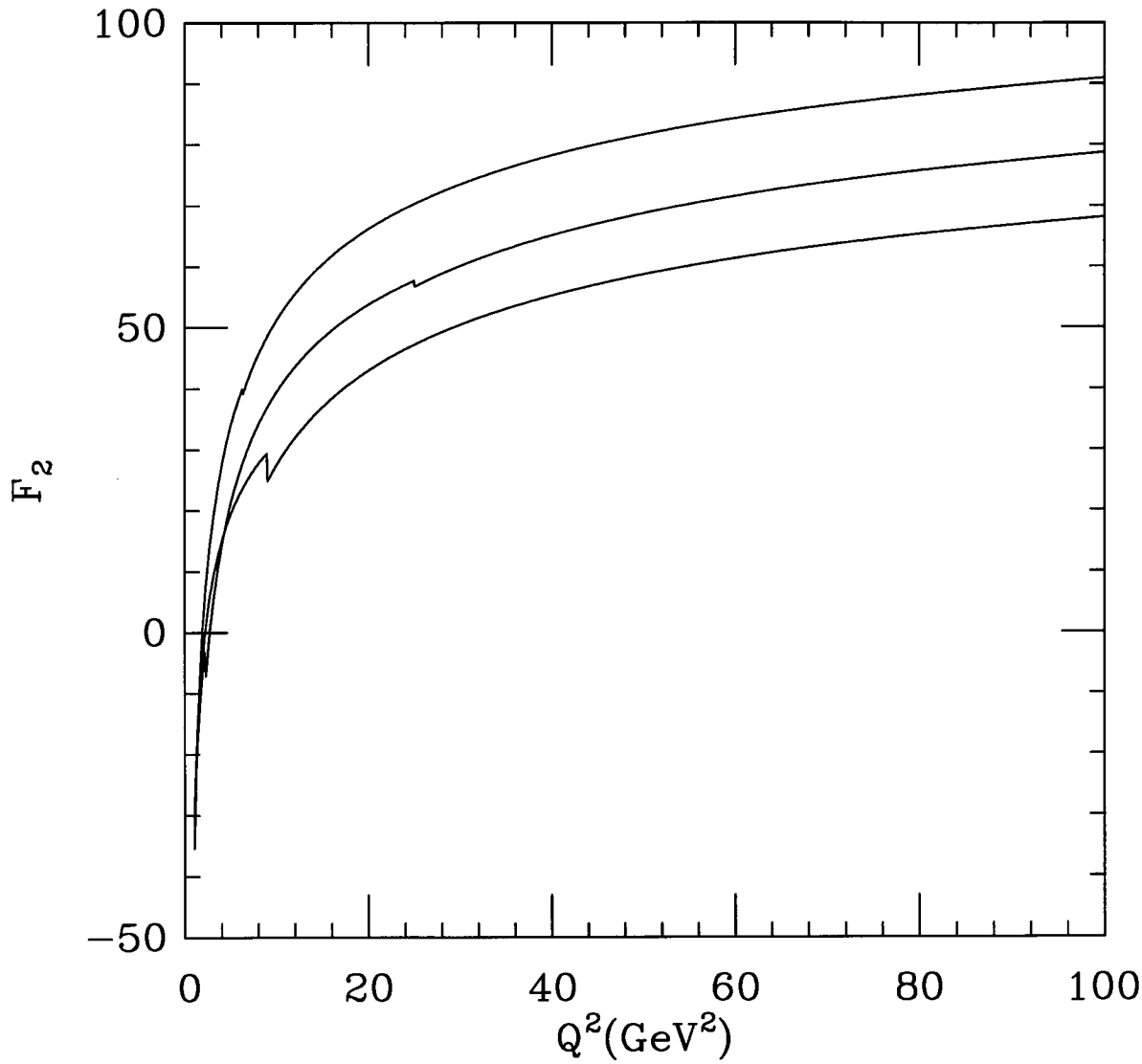

Figure E.4.2: F_2 cross section in the ZM-VFNS for $\eta = 0.01$

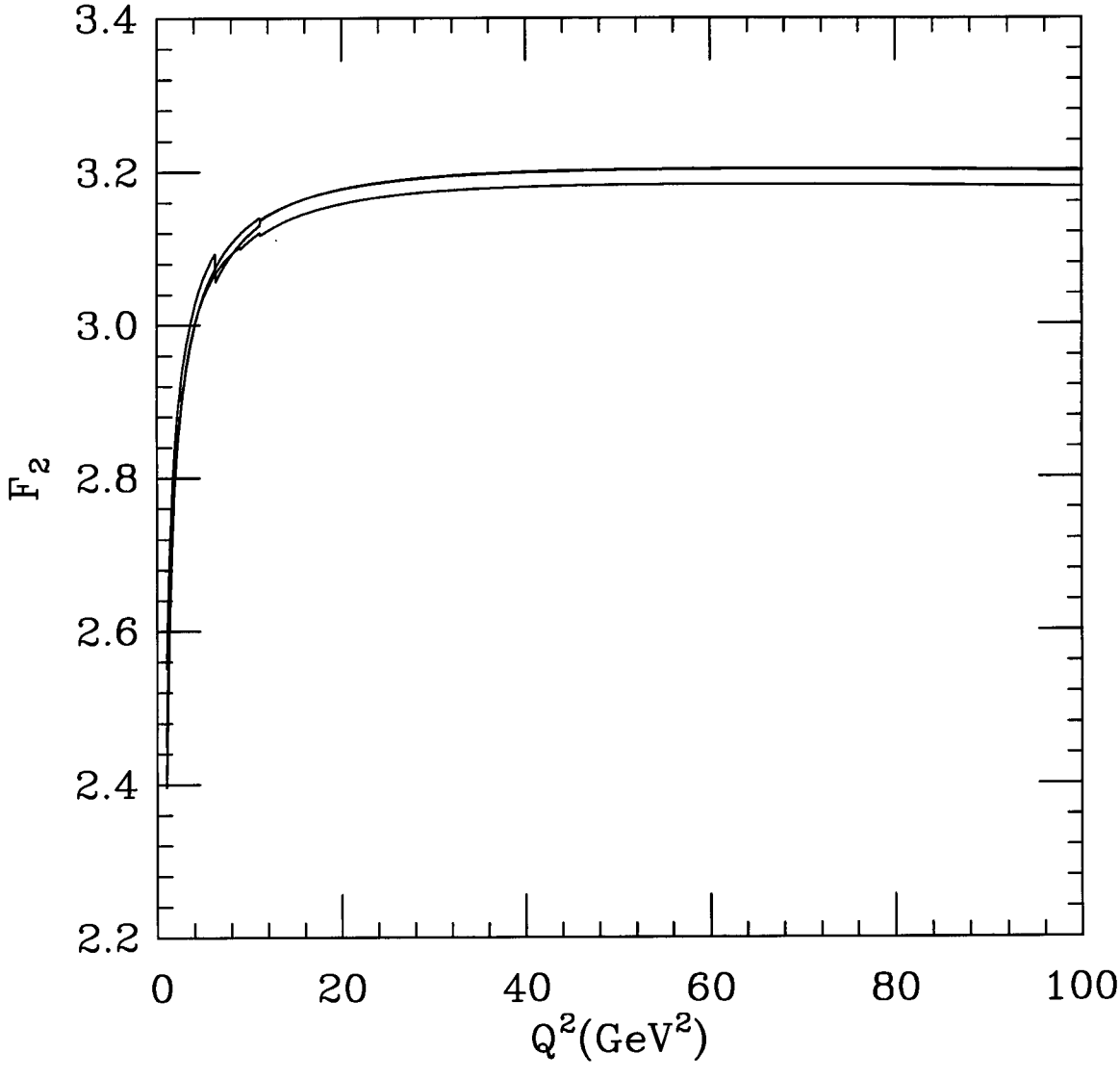

Figure E.4.3: F_2 cross section in the VFNS for $\eta = 0.1$

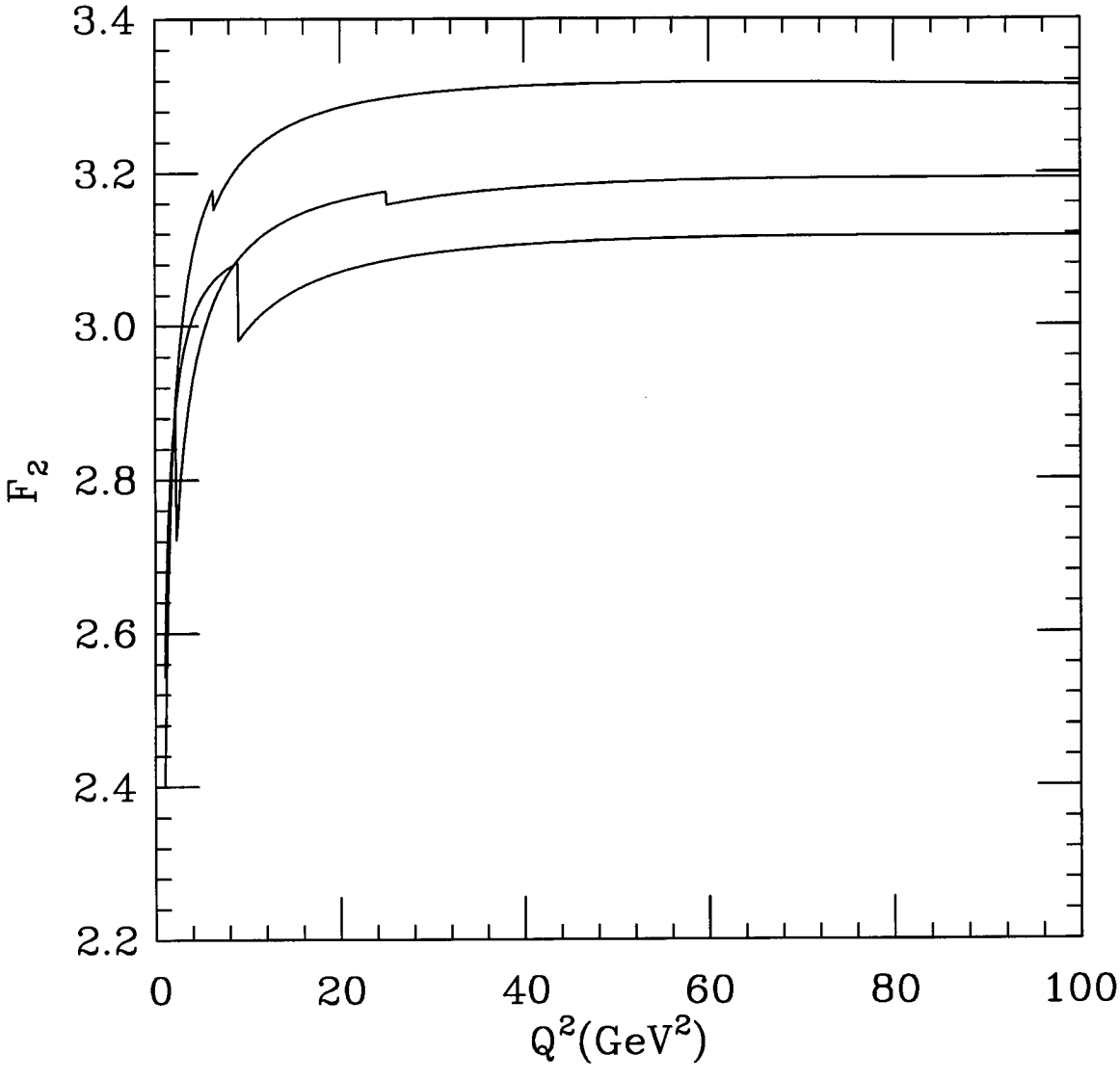

Figure E.4.4: F_2 cross section in the ZM-VFNS for $\eta = 0.1$

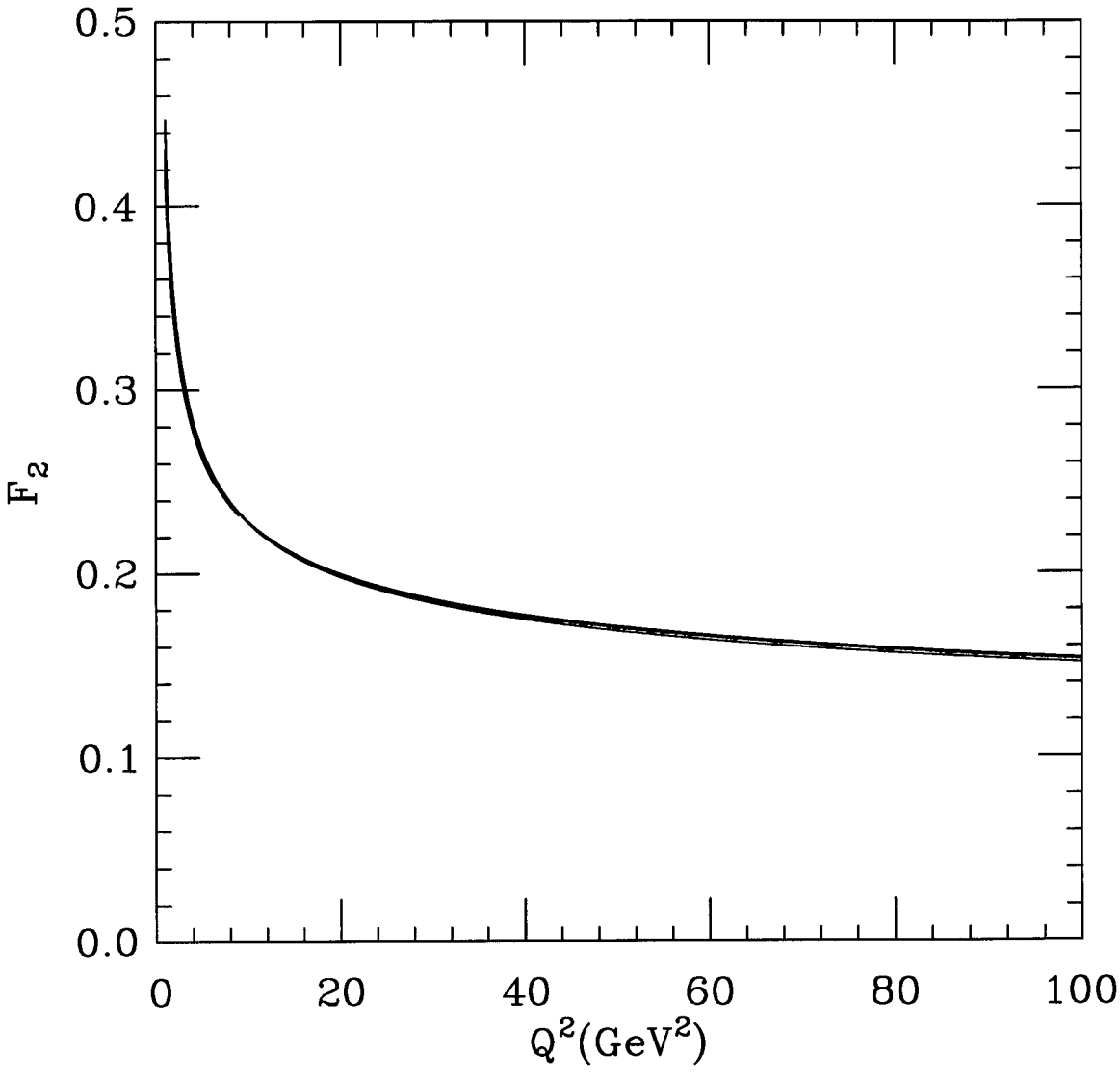

Figure E.4.5: F_2 cross section in the VFNS for $\eta = 0.5$

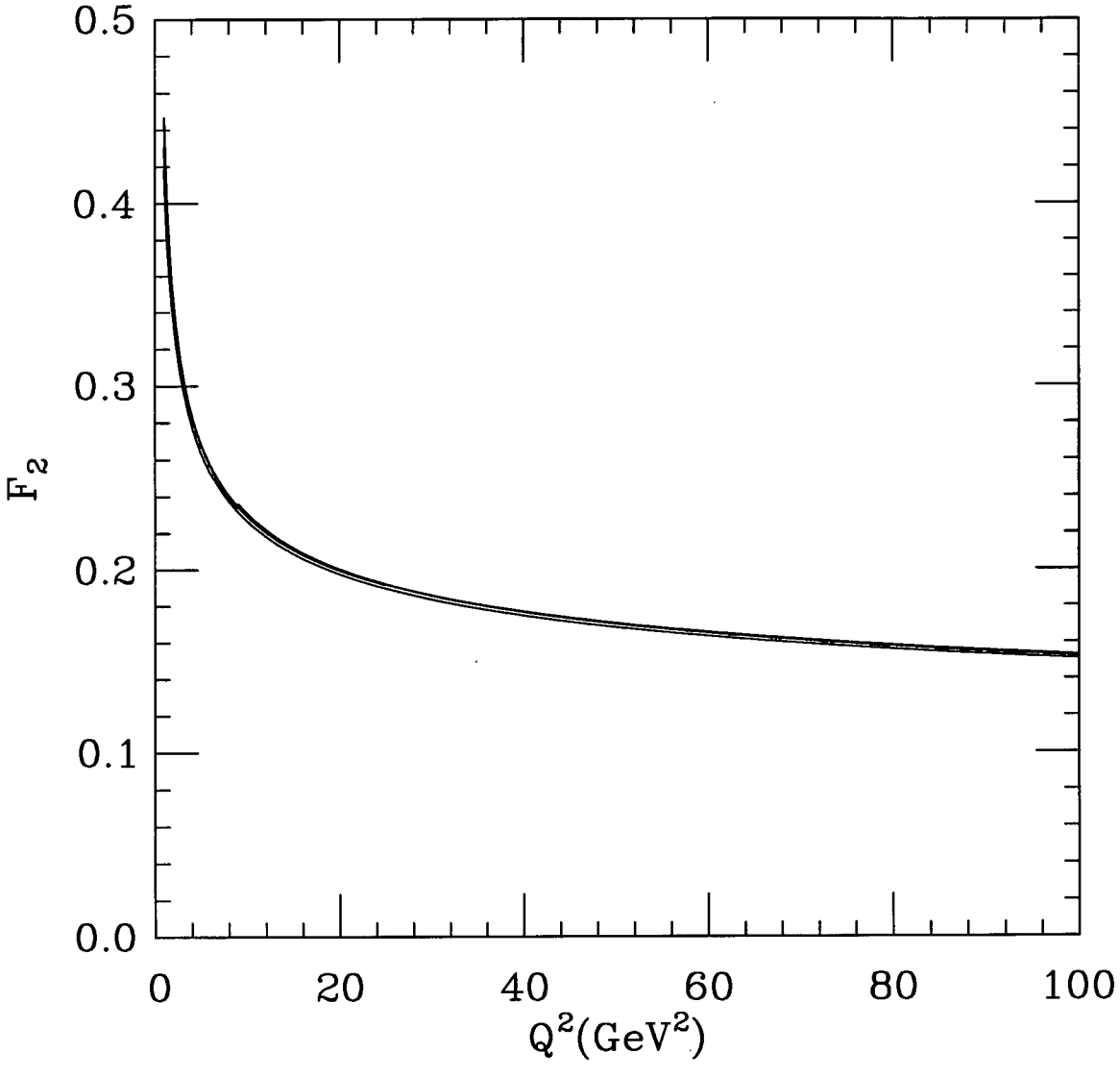

Figure E.4.6: F_2 cross section in the ZM-VFNS for $\eta = 0.5$

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